QUALIFYING EXAM

Sept. 2007

Note: *Show all work to receive full credit. Calculators are NOT allowed!*

1. Let $f(t) = \int_t^{t^3} ln(e^x + 1)\, dx$. Find $f'(t)$.

2. Find the coefficient of x^n in the Maclaurin expansion of

$$f(x) = \frac{1}{(1-2x)(1+x)},$$

and find the range where this series converges. *Hint:* Apply partial fraction decomposition.

3. Evaluate the sum/limit of the following series in closed form.

(a)

$$\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$$

(b)

$$\sum_{n=1}^{\infty} n^2 \left(-\frac{1}{3}\right)^n$$

4. (a) Find the matrix A for the transformation $T : R^2 \to R^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2(x+y) \\ 5y - x \end{bmatrix}$$

using the standard basis $\{\vec{e_1}, \vec{e_2}\} = \{[1,0]', [0,1]'\}$.

(b) Find the eigenvalues and matching eigenvectors of A.

5. Let $\vec{v} \mapsto P(\vec{v})$ be an orthogonal projection onto the plane L with equation $2x - y + 2z = 0$ in R^3. Find the matrix of P with respect to the usual orthonormal basis $\vec{i}, \vec{j}, \vec{k}$.

6. A three-dimensional region is described by

$$R = \left\{(x,y,z) : 4x^2 + 9y^2 - z^4 \le 1,\ 0 \le z \le 1\right\}.$$

(a) Find the volume of R. *Hint:* The area of an ellipse with semi-major axis length a and semi-minor axis length b is πab.

(b) Find the z-coordinate of the centroid of R.

7. The sequence $a_0, a_1, a_2, a_3, \ldots = 1, -4, 10, -34, \ldots$, respectively, satisfies the recursion relation

$$a_{n+2} = -a_{n+1} + 6a_n, \qquad n = 0, 1, 2, \ldots \text{ with } \; a_0 = 1 \text{ and } a_1 = -4.$$

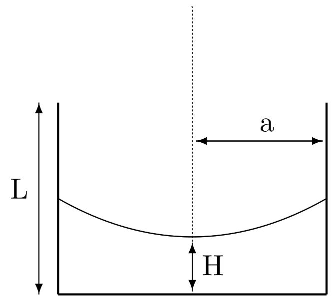
In terms of the non-negative integer k, what is a_k? If it exists, what is $L = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$?
Hint: Try $a_k = \lambda^k$ as a trial solution of the recursion relation without initial conditions. Any linear combination of such solutions is also a solution.

8. If $y'' - 4y' + 4y = e^{2x}$, $y(0) = 1$, $y'(0) = 0$, find y as a function of x.

9. Water partially fills a cylindrical pail of radius a and height L. When the pail is rotated about its symmetry axis with angular speed Ω, the surface of the water assumes a parabolic shape the profile of which is given by

$$z = H + \frac{\Omega^2}{2g} r^2, \quad r \leq a,$$

where z is the height of the surface, r is the distance from the symmetry axis, and H is the surface height at the axis. See the figure below.



The volume of water in the pail is fixed at amount V.
(a) Give the equation that relates H to Ω. Note that this equation does not contain the variables z and r. Solve this equation explicitly for H and for Ω.
(b) At what value of Ω does the water surface touch bottom?
(c) Suppose Ω is gradually increased from 0 until the water flows over the rim of the pail before its surface touches the bottom. At what value of Ω does the water start to flow over the rim? (Assume in this case that the height of the pail L is short enough so that the water surface does *not* touch the bottom when the water flows over the rim.)

10. The point Q moves at constant speed v counterclockwise around a circle of radius R centered at the origin O. The point P starts at the origin at time $t = 0$ and moves at constant speed v directly toward Q. Does P ever catch Q? If so, how long does it take and through what angle will the ray $\overrightarrow{OQ}$ have moved during the "chase"? Express your answers in terms of R and v.