# Qualifying Exam
September 2009

You may use calculators for this exam. Show all work to receive full credit.

1. The following figure shows a rope running through a pulley at P, bearing a weight W at one end. The other end of the rope is at point M, where it is being pulled along the ground, away from the pulley, at a rate of 6 ft/sec. Suppose that P is 25 feet above the ground and that the rope is 45 feet long. How fast is the weight being raised at the instant when the distance x is 15 feet?
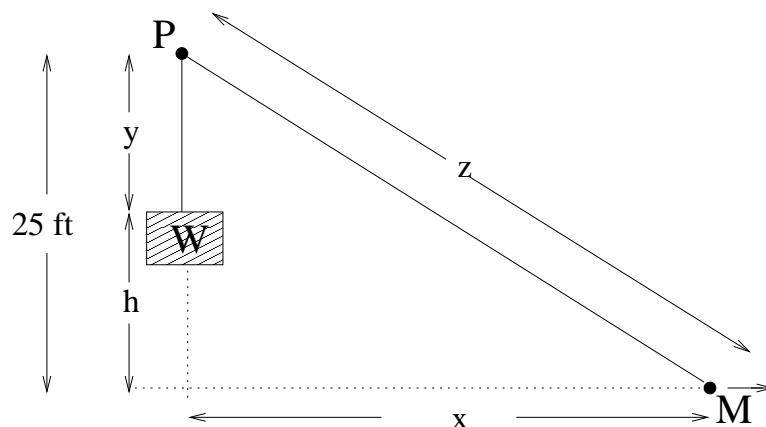
2. Consider the following figure showing traffic patterns with all measurements given in cars per minute. Assuming that none of the flows can be negative, what is the smallest possible value for x_3?
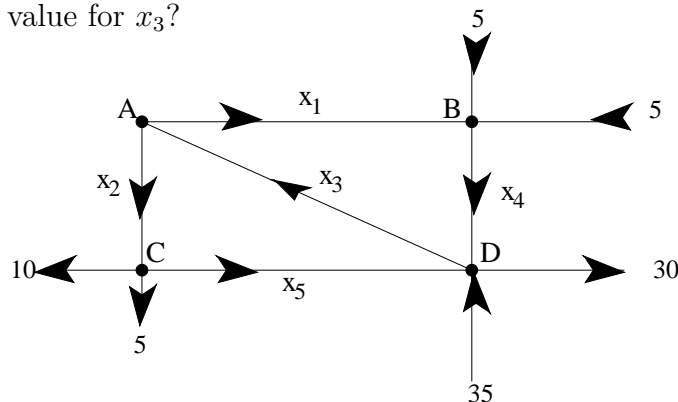
3. Find the mass of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ if the density at any point is proportional to the distance from the point to the z-axis and the density at point $(1,1/2, 5/4)$ is $3\sqrt{5}/4$.

4. Find the maximum value of $f(x, y, z) = xyz$ among all points (x, y, z) lying on the line of intersection of planes $x + y + z = 30$ and $x + y - z = 0$.

5. Each year 1/10 of the people outside Metropolis move to Metropolis and 2/10 of the people inside Metropolis move out. At time 0 there are 10 million people living in Metropolis and 200 million living outside.

(a) Determine a 2×2 matrix A so that the vectors $\begin{bmatrix} m_k \\ o_k \end{bmatrix}$ of people living inside and outside Metropolis in year k are related by $\begin{bmatrix} m_{k+1} \\ o_{k+1} \end{bmatrix} = A \begin{bmatrix} m_k \\ o_k \end{bmatrix}$.

(b) Express the population vector in year 0 as a linear combination of the eigenvectors of A.

(c) Determine the number of people living in Metropolis in year k as a function of k. What will happen to the population of Metropolis eventually?

6. Let $p_0, p_1, p_2, \ldots$ be a sequence of real numbers. For any function $g : \mathbb{R} \to \mathbb{R}$, define a function T such that $T(g(x)) = \sum_{x=0}^{\infty} g(x) p_x$. Suppose $p_x = \frac{\lambda^x}{x!}$, where λ is a positive constant. Derive closed-form expressions for $T(x)$ and $T(x^2)$.

7. Torricelli's Law states that the rate of flow of fluid out of a container through a hole in the bottom is proportional to the square root of the depth of fluid above the hole. Suppose that a container has the shape of a cone (point downward) with radius 5 feet and height 10 feet, that it is full of water initially, and that the initial rate of flow is 1 ft^3/sec. (It may help to remember that the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.)

(a) Determine an expression for the depth of water in the container at time t.

(b) For what values of t does the expression represent the depth of water? Explain what the expression says about the depth of water over time and why you chose the range of values of t that you did.

8. Consider a curve in the first quadrant of $\mathbb{R}^2$ with an interesting property: If you pick any point, $P1$, on the curve, and let $P2$ be the x-intercept of the line tangent to the curve at $P1$, then the y-axis will bisect the line segment from $P1$ to $P2$. The curve contains the point $(1,2)$. Derive the equation of the curve.

9. Let S be the surface $xy^2 z + x^2 yz^2 = 30$.

(a) Find the equation of the plane tangent to S at $(3,2,1)$.

(b) Use your answer to (a) to determine in what direction from $(3,2,1)$ you should go in order to reach the surface $xy^2 z + x^2 yz^2 = 29$ in the shortest distance. Estimate this distance. Explain your answers briefly, using a diagram.

10. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $x^2 + 4y^2 + 9z^2 = 9$.