Qualifying Exam

September 13, 2010

Directions: In all problems you must show your work in order to receive credit. You may use a calculator BUT you must explain thoroughly how you obtained your answers!

1. A spotlight on the ground shines on a wall 12 meters away. If a man 2 meters tall walks from the spotlight toward the wall at a speed of 1.6 meters per second, how fast is the length of his shadow on the building decreasing when he is 4 meters from the wall?

   Solution:

   Using the similar triangles in the diagram above, we get the ratio of sidelengths $\frac{x}{2} = \frac{12}{y}$, or $y = \frac{24}{x}$. We are given that $\frac{dx}{dt} = 1.6$ m/s, and wish to find $\frac{dy}{dt}$. Differentiating, we have
   $$\frac{dy}{dt} = -\frac{24}{x^2}\frac{dx}{dt} = -\frac{24}{64}(1.6) = -0.6 \text{ m/s}.$$
   The length of the shadow is decreasing at a rate of 0.6 m/s.

2. Let $\alpha \in \mathbb{R}$ and
   $$f(x) = \begin{cases} \frac{1}{n^\alpha} & \text{if } x = \frac{1}{n},\ n \text{ a positive integer,} \\ 0 & \text{for all other real numbers.} \end{cases}$$
   Determine, using just the definitions involved, the values of α such that

   (a) f is continuous at $x = 0$.
   (b) f is differentiable at $x = 0$.

   Solution: For (a), we must have $\lim_{x\to 0} f(x) = f(0) = 0$. When $x \neq 1/n$, we have $f(x) = 0$, so α must be such that $f(1/n) = 1/n^\alpha \to 0$ as $n \to \infty$. Therefore we must have $\alpha > 0$. For (b), we need for
   $$\lim_{h\to 0} \frac{f(h)}{h}$$
   to exist. Hence we need
   $$\frac{f(1/n)}{1/n} = \frac{1}{n^{\alpha-1}} \to 0$$
   as $n \to \infty$. This occurs when $\alpha > 1$.

3. A bowl is shaped like a hemisphere with diameter 30 cm. A ball with diameter 10 cm is placed in the bowl and water is poured into the bowl to a depth of h cm, so that the top of the ball is above the water level. Assuming the ball does not float, find the volume of water in the bowl.

   Solution:

   We build an integral using horizontal cross sections, each of which is an annulus with height Δy. Hence the integral will be of the form

   $$\int_0^h (\pi r_1^2 - \pi r_2^2)\, dy,$$

   where r_1 and r_2 are the outer and inner radii, respectively. Applying the Pythagorean Theorem to the triangles in the diagram, we find that $r_1^2 = 15^2 - (15-y)^2$, and $r_2^2 = 5^2 - (y-5)^2$. Note that due to the squaring, it does not matter whether the height y is above or below the center of the ball. We then have:

   $$\pi \int_0^h [15^2 - (15-y)^2] - [5^2 - (y-5)^2]\, dy = \pi \int_0^h 20y\, dy = 10\pi h^2.$$

4. Find the minimum and maximum values of $f(x,y) = 6xy - 4x^3 - 3y^2$ on the closed triangular region bounded by the lines $y = 0$, $x = 1$ and $y = 3x$.

   Solution: The partial derivatives of f are $\frac{\partial f}{\partial x} = 6y - 12x^2$ and $\frac{\partial f}{\partial y} = 6x - 6y$. Setting each equal to 0, we solve for the critical points, which are $(0,0)$ and $(1/2, 1/2)$, and yield values $f(0,0) = 0$ and $f(1/2,1/2) = 1/4$. Next we optimize f on the boundary triangle. On the line segment $y = 0$, $0 \le x \le 1$, $f(x,0) = -4x^3$, which has its maximum at $f(0,0)$ and its minimum at $f(1,0) = -4$. On $x = 1$, $0 \le y \le 3$, we have $f(1,y) = 6y - 4 - 3y^2$. There is a critical point at $(1,1)$, and $f(1,1) = -1$. We also check the endpoint $f(1,3) = -13$. Finally, on $y = 3x$, $0 \le x \le 3$, we have $f(x,3x) = -9x^2 - 4x^3$. There are critical points of this function at $x = 0$ and $x = -3/2$. The first has already been evaluated, and the second is outside the boundary. We have now checked all necessary points. Comparing the value of f at the points $(1/2,1/2)$, $(0,0)$, $(1,0)$, $(1,1)$ and $(1,3)$, we find that the maximum value of $f(x,y)$ is $1/4$, and the minimum value is -13.

5. Compute $\displaystyle\int_0^2 \int_{y^2}^4 y\cos(x^2)\ dxdy$.

Solution:

The region of integration is drawn above. Since the integral can not be evaluated as given, we change the order of integration:

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)\ dxdy = \int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)\ dydx = \frac{1}{2}\int_0^4 x\cos(x^2)\ dx = \frac{1}{4}\sin(u)|_0^{16} = \frac{1}{4}\sin(16),$$

where the substitution $u = x^2$ is made.

6. (a) Find the Maclaurin series (Taylor series at 0) for $\sin x$ and $\cos x$.

(b) Use part (a) to find real numbers A and B such that

$$\lim_{x\to 0} \frac{A\sin x - x(1 + B\cos x)}{x^3} = 1.$$

Solution: For (a), we have

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

and

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots .$$

Then for (b) we want

$$\lim_{x\to 0} \frac{A(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) - x - Bx(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots)}{x^3} = 1.$$

The terms on the left with positive power of x will go to 0 in the limit, so we must have

$$\lim_{x\to 0} \frac{A - 1 - B}{x^2} - \frac{A}{6} + \frac{B}{2} = 1.$$

Hence we need $A = B + 1$ and $-\dfrac{A}{6} + \dfrac{B}{2} = 1$. Solving this system of equations yields $A = 9/2$ and $B = 7/2$.

7. Assume as known that $1 + x \le e^x$ for $x \ge 0$. Define the sequence $(a_n)_{n\ge1}$ recursively by

$$a_0 = 1/2, \quad a_n = \ln(1 + a_{n-1}) \text{ for } n \ge 1.$$

(a) Prove that $\lim\limits_{n\to\infty} a_n = 0$.

(b) Find the radius of convergence of the power series $\sum\limits_{n\ge0} a_n x^n$, where $(a_n)_{n\ge1}$ is the sequence defined above.

Solution: For (a), we have $a_n > 0$ for all n since $a_0 > 0$ and $\ln x$ is positive for $x > 1$. Since $1+a_n < e^{a_n} = 1+a_{n-1}$, we see that (a_n) is decreasing. Therefore by the Monotone-Bounded Convergence Theorem, the sequence has a limit, L, as $n \to \infty$. Then passing to the limit in the recursive formula, we have $L = \ln(1 + L)$, hence $e^L = 1 + L$. The only solution to this equation is $L = 0$, since the line $y = 1 + x$ is tangent to the curve $y = e^x$ at $x = 0$, and $y = e^x$ is concave up on its domain. Therefore, $\lim\limits_{x\to0} a_n = 0$.

For (b), we use the Ratio test. We want:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \frac{\ln(1 + a_n)}{a_n} = |x| \lim_{n\to\infty} \frac{1}{1 + a_n} = |x| \cdot 1 < 1,$$

by using l'Hopital's Rule and part (a). Therefore the radius of convergence is 1.

8. Let $a, b \in \mathbb{R}\backslash\{0\}$. Consider the symmetric matrix

$$A = \begin{bmatrix} 1 & a & b \\ a & a^2 & ab \\ b & ab & b^2 \end{bmatrix}.$$

(a) Find the kernel and the rank of A.

(b) Find two orthogonal eigenvectors for A.

Solution: Via row operations we have

$$A \sim \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the kernel of A consists of vectors of form $[x_1\ x_2\ x_3]^T$ where $x_1 = -ax_2 - bx_3$. Hence

$$\text{Ker}(A) = \text{span}\left(\left\{ \begin{bmatrix} -a \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -b \\ 0 \\ 1 \end{bmatrix} \right\}\right).$$

Then since the kernel of A has dimension 2, the rank of A is $3 - \dim(\text{Ker}(A)) = 1$.

By part (a), the basis given for the kernel gives two linearly independent eigenvectors corresponding to the eigenvalue $\lambda = 0$. Let $v_1 = [-a\ 1\ 0]^T$ and $v_2 = [-b\ 0\ 1]^T$. To make this set orthogonal, we replace v_1 by

$$v_1 - \frac{(v_1 \cdot v_2)v_2}{\|v_2\|^2} = \begin{bmatrix} -\frac{a}{b^2+1} \\ 1 \\ -\frac{ab}{b^2+1} \end{bmatrix}.$$

Rescaling we have

$$\left\{ \begin{bmatrix} -a \\ b^2+1 \\ -ab \end{bmatrix}, \begin{bmatrix} -b \\ 0 \\ 1 \end{bmatrix} \right\}.$$

9. Let V be a 3-dimensional vector space over a field F, and $T : V \to V$ a linear transformation with $T^2 \neq 0$ but $T^3 = 0$, where $0 : V \to V$ is the zero map.

   (a) Show that there is an element $v \in V$ such that $\{v, T(v), T^2(v)\}$ is a basis for V.

   (b) Is T diagonalizable? (Think about possible eigenvalues of T.)

   Solution: For (a), since $T^2 \neq 0$, there is some $v \in V$ with $T^2(v) \neq 0$. We will show that for such a v, $\{v, T(v), T^2(v)\}$ is a basis for V. Since it has three elements, we only need to show that the set is linearly independent. Suppose that $av + bT(v) + cT(v)^2 = 0$ for some $a, b, c \in F$. Applying T to both sides and using linearity yields $aT(v) + bT(v^2) = 0$, since $T^3 = 0$. Applying T once more yields $aT^2(v) = 0$, and hence $a = 0$. Making this substitution above, we see that $b = 0$ and hence also $c = 0$. Therefore the set is a basis.

   For (b), the matrix for T, using the ordered basis $\{v, T(v), T^2(v)\}$, is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

   so the only eigenvalue is 0. The kernel of M consists of vectors of the form $[x_1\ 0\ 0]^T$, so it has dimension 1. In order for T to be diagonalizable, the sum of the dimensions of the eigenspaces must equal 3. Therefore T is not diagonalizable.

10. Is there a differentiable function on $[0, \infty)$ whose derivative equals its 2010th power and whose value at the origin is strictly positive?

    Solution: Such a function must satisfy the differential equation $y' = y^{2010}$ with $y(0) = k > 0$. This is a separable equation, which can be solved by:

$$\begin{aligned} y^{-2010}\, dy &= dx \\ y^{-2009} &= -2009x + C, \end{aligned}$$

where $C = k^{-2009}$. Then solving for y:

$$y(x) = \frac{1}{(k^{-2009} - 2009x)^{1/2009}}.$$

Since $k > 0$, $y(x)$ has a vertical asymptote at $x = k^{-2009}/2009$. Therefore y can not be differentiable. Hence there is no such function.