Graduate Qualifying Exam – Fall 2014

Instructions: **Write your identifying number on each page you turn in**. Each problem is worth 10 points. In all problems you must show your work and explain your answers in order to receive credit. You may use calculators for this exam. Be advised however that every question can be answered without the use of a calculator and more than likely can be answered more efficiently without the use of a calculator. You have **three hours**.

Problem 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 2$ and $|f(x) - f(y)| \leq |x - y|^{5/4}$ for all real numbers x and y.

(a) (4 points) Prove that f is continuous at all x using the rigorous $\epsilon - \delta$ definition of continuity.

(b) (4 points) Prove that f is differentiable at all x using the definition of the derivative.

(c) (2 points) Compute $\displaystyle\int_3^6 f(y)\, dy$.

Problem 2. Let $f(x) = \displaystyle\int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\, dt$, where $g(x) = \displaystyle\int_0^{\cos x} (\sin(t^2) + 1)\, dt$. Find $f'(\pi/2)$.

Problem 3. Find the point on the parabola $y = 1 - x^2$ in the first quadrant at which the tangent line together with the x- and y- axes forms the triangle with smallest area.

Problem 4. Find all points P on the ellipsoid $2x^2 + 2y^2 + z^2 = 28$ such that the tangent plane to the ellipsoid at P is parallel to the plane passing through the the three points $(1, 3, 1)$, $(3, 0, -3)$, and $(0, 4, 2)$.

Problem 5. Find the volume of the smaller of the two regions enclosed by the surfaces $z = 1 + x^2 + y^2$ and $x^2 + y^2 + z^2 = 11$.

Problem 6. Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)} \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz = 2\pi.$$

(Hint: the integral above is improper. To evaluate it, you should compute a triple integral over a suitably chosen bounded region and take the limit as that region grows without bound.)

Problem 7. Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in a vector space V. Working directly from the definition of linear independence, show that $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}\}$ is also linearly independent.

Problem 8. Let M be the 1000×1000 matrix consisting of all 1s. Find the characteristic polynomial for M.

Problem 9. Consider the differential equation $y' = Ay^2$ where A is a real constant.

(a) (5 points) Find the general solution (your solution will contain the parameter A).

(b) (5 points) Find a value of A for which there exists a solution $y(t)$ that satisfies $y(0) > 0$ and $y(1) < 0$ and is continuous on an open interval containing the closed interval $[0, 1]$ or explain why such an A does not exist.

Problem 10. For $n \in \mathbb{N}$, define $a_n = \sqrt{n+1} - \sqrt{n}$.

(a) (4 points) Compute $\lim\limits_{n\to\infty} a_n$.

(b) (6 points) Does the series $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ converge absolutely, converge conditionally or diverge? Justify your answer by using one or more series tests, making sure to explain why the tests apply.