Qualifying Examination

15 September 2008

Explain all your answers and show all your work.
Calculators are **not** permitted.
Time allowed: 3 hours.

1. Solve the differential equation $\frac{dy}{dx} = 2y + x + e^x$ with initial condition $y(0) = 1$.

2. You are climbing a mountain by the steepest rate at a slope of 20° when you come to a trail branching off at a 30° angle from yours. What is the angle of ascent of this branch trail?

3. The bottom of a waffle ice cream cone forms a wide cone with an angle of 60° with its vertical axis and is filled with ice cream. If the top of the ice cream in the cone forms part of a hemisphere of radius 3 inches centered at the apex of the cone, what is the total volume of ice cream in the cone?

4. A South American pyramid has a square base of side 100 ft and is 100 ft high. If the stones making up the pyramid weigh 60 lbs/ft^3, set up and evaluate an integral for the total work done in constructing the pyramid.

5. The theorem of Pappus says that the volume V of a solid formed by rotating a planar region A about a line in the plane not intersecting A is given by the area $|A|$ of A multiplied by the distance d traveled by the centroid (center of mass) of A during the rotation. (In symbols: $V = d|A|$.)

   (a) Prove Pappus' theorem for the region between two functions $f(x) \le g(x)$ where $0 \le a \le x \le b$.

   (b) Use Pappus' theorem to find the center of mass of a semicircle of radius 1, centered at the origin.

6. Find $y'(x)$ if $x > 0$ and $y(x) = \int\limits_{\ln x}^{x} e^{(x+t)^2} dt$.

7. Define $y(x) = \left(x^2\right)^x$ for $x \neq 0$.

   (a) Find $\lim_{x\to 0} \left(x^2\right)^x$.

   (b) Find the maximum and minimum values of $\left(x^2\right)^x$ on the interval $[-1, 1]$.

8. (a) Find the interval of convergence of $f(x) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n^2}$. Show your work in detail.

   (b) Using integration or differentiation or some combination of these, find a closed form representation for the series $g(x) = \sum\limits_{n=2}^{\infty} \frac{x^{n+1}}{n(n-1)}$ (where it converges).

9. Newton's Law of Cooling says that the rate of change of the temperature of a body is proportional to the difference between its temperature and that of the surrounding medium. A vat of boiling soup at 100°C is brought into a room where the air is 20°C, and left to cool. After 1 hour, its temperature is 60°C. How much additional time is required for it to cool to 30°C?

10. Let $\mathbf{F}$ be the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

    (a) Find a formula for $\mathbf{F}^n$.

    (b) Hence, or otherwise, find a formula for the n^{th} Fibonacci number. (The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$.)