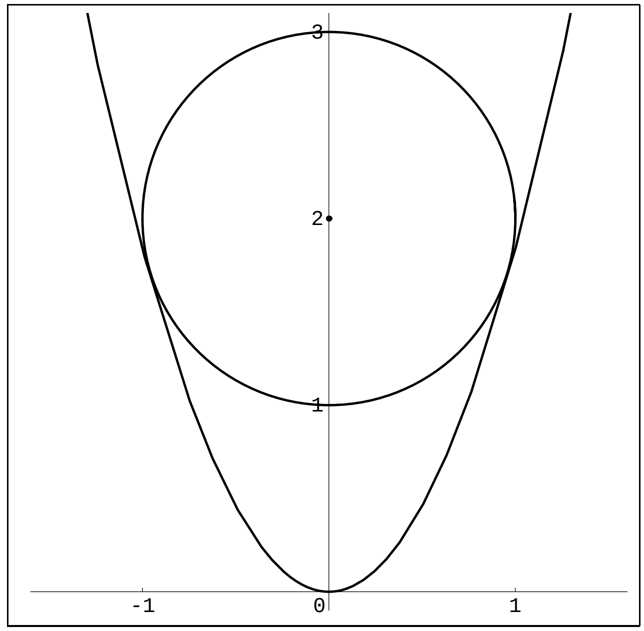
Qualifying Exam

Spring 2007
April 2, 2007

Number ________________

1. The figure on the right shows a circle of radius 1 centered at $(0,2)$ inscribed in a parabola $y = ax^2$. Find the exact value of a.



2. Let $b > 0$. Consider the graph of the function $y = e^{bx}$.

(a) Write the equation of the tangent line to the graph of $y = e^{bx}$ at the y-intercept.

(b) Write the equation of the tangent to the graph of $y = e^{bx}$ that passes through the origin.

(c) Find $b > 0$ which maximizes the acute angle between the tangent lines found in (2a) and (2b).

3. (a) Disprove the following statement:

If $\displaystyle\sum_{n=1}^{+\infty} a_n$ converges, then the series $\displaystyle\sum_{n=1}^{+\infty} a_n^2$ converges.

(b) Prove the following statement:

If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\displaystyle\sum_{n=1}^{+\infty} a_n$ converges, then the series $\displaystyle\sum_{n=1}^{+\infty} a_n^2$ converges.

4. In this problem we use the floor and the ceiling function. These functions are defined by

$$\operatorname{floor}(x) = \lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}, \quad \operatorname{ceiling}(x) = \lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}, \quad x \in \mathbb{R}.$$

Notice that the following inequalities hold:

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1, \qquad \lceil x \rceil - 1 < x \leq \lceil x \rceil, \quad x \in \mathbb{R}.$$

For each of the following integrals determine whether it converges or diverges. Give a detailed explanation of your reasoning. If you claim convergence calculate the exact value of the integral.

$$\text{(A)} \quad \int_0^{+\infty} \left\lfloor \frac{1}{x} \right\rfloor dx; \qquad\qquad \text{(B)} \quad \int_0^1 \left(\left\lceil \frac{1}{x} \right\rceil - \left\lfloor \frac{1}{x} \right\rfloor \right) dx.$$

5. Find (x_0, y_0) in the set $\{(x,y) \in \mathbb{R}^2 \text{ such that } x \leq y\}$ where the function

$$f(x,y) = \int_x^y \left(1 - e^{1-s^2}\right) ds$$

attains its minimum.

OVER

6. (a) In absence of other factors (e.g. birds), the population of mosquitoes in a certain area increases at a rate proportional to the current population size. It is observed that the population doubles in 10 days. Write the differential equation governing the growth of the population of mosquitoes.

(b) It is suggested that a fixed number of predator birds should be brought to the area to eliminate mosquitoes. Assume that one bird eats mosquitoes at the rate of 5000 mosquitoes per day. Modify the differential equation from (6a) to model this new situation.

(c) It is estimated that there are 10 million mosquitoes in this area. Determine the minimum number of birds that should be brought to the area in order to eliminate the mosquito population. Give the exact symbolic answer and the best estimate that you can find. Give all details of your reasoning.

7. Let $Z = \{(x, y, z) \in \mathbb{R}^3 \text{ such that } 0 < z \leq 1, x^2 + y^2 \leq z^4\}$.

(a) Sketch the set Z.

(b) Evaluate the triple integral

$$\iiint_Z \left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2\right) dV.$$

8. (a) Let A be a 3×3 matrix. Give a definition of an eigenvalue of A.

(b) Prove that the matrices given below have the same eigenvalues.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 4 & -3 & -6 \\ 0 & -1 & -4 \\ 1 & -1 & -1 \end{pmatrix}$$

9. Find all value(s) of $t \in \mathbb{R}$ for which the matrix

$$\begin{pmatrix} t-1 & 2+t-t^2 \\ 1+t & -3-t \end{pmatrix}$$

has a repeated eigenvalue for which the corresponding eigenspace is one-dimensional.

10. The contour diagrams on the next page show contour lines at indicated levels. We used six out of the following eight functions

(A)	$z = \dfrac{y}{1+x^2}$	(B)	$z = \sqrt{2-(y-x)}$	(C)	$z = \dfrac{e^y}{e^x}$	(D)	$z = x + \sqrt{y}$
(E)	$z = \dfrac{e^x}{e^y}$	(F)	$z = \sqrt{4-(x-y)^2}$	(G)	$z = \dfrac{y^2}{1+x^2}$	(H)	$z = \sqrt{2-(x-y)}$

Identify which contour diagram belongs to which function. Provide a brief explanation for your choice. Place the letter corresponding to the function in the box with its contour diagram.

OVER

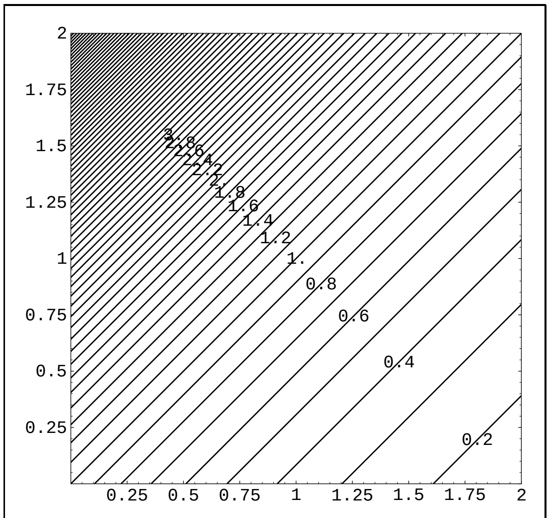


Figure 1.

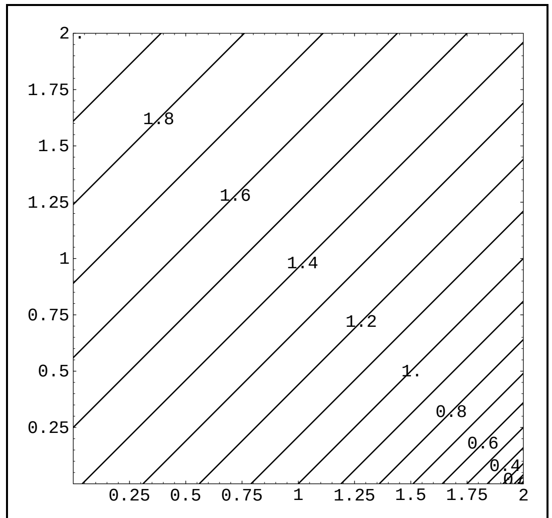


Figure 2.

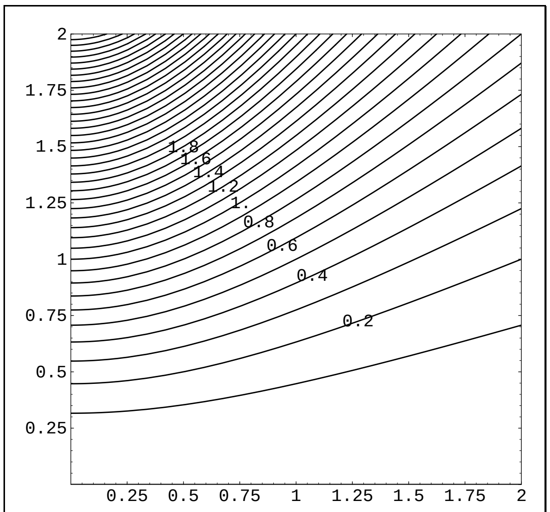


Figure 3.

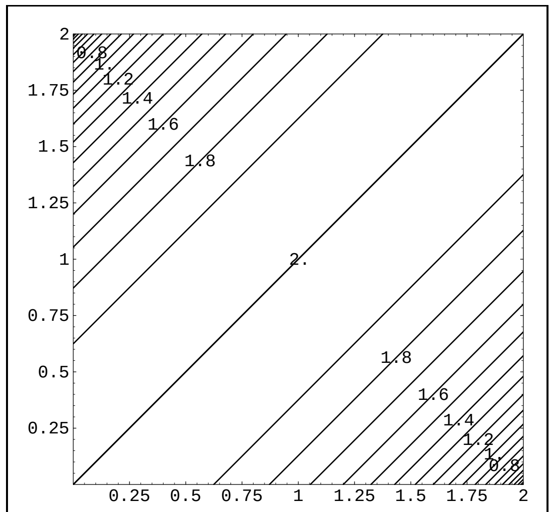


Figure 4.

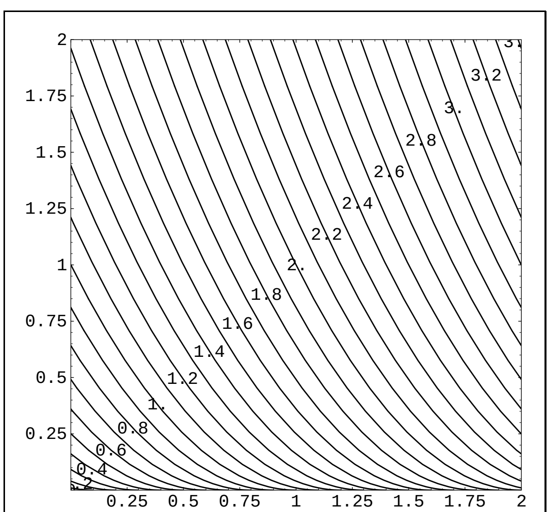


Figure 5.

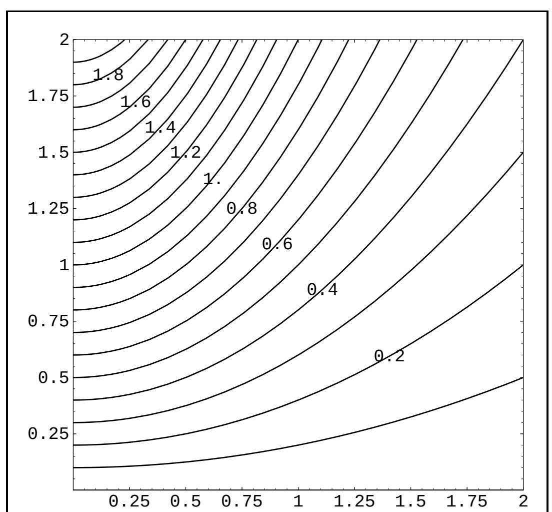


Figure 6.