Qualifying Exam – Fall 2012

September 17, 2012

- The allocated time for this exam is 240 minutes.
- Write your identifying number on every page you use.
- All problems have equal weight.
- **Calculators are permitted. Show your work for full credit.**

1. Find $y'(x)$ if $y(x) = \displaystyle\int_1^{x^2} \frac{\sin(tx)}{t}\, dt$.

2. A ball has a radius of 0.5 meters and weighs 10 kg. It is floating in a large pool of water. How deeply is it submerged at its deepest point? (Recall the Archimedean principle: For a floating object, the weight of the displaced water equals the weight of the object. The density of water is approximately 1000 kg/m^3.)

3. Consider three tanks, labeled A, B and C. Initially, tank A contains 20 gal of water with a salt concentration of 0.2 lb/gal. Both tanks B and C initially contain 10 gal of pure water without any salt. Water is flowing from tank A to tank B at the rate of 0.1 gal/min, from tank A to tank C at the rate of 0.1 gal/min, and from tank B to tank C at the rate of 0.2 gal/min. In addition, water is leaking out of tank C at the rate of 0.1 gal/min. Assume perfect mixing, that is, the salt solution mixes effectively instantly in the tanks.

   What is the salt concentration in tank C when its maximum capacity of 20 gal is reached?

4. The temperature at any point of a flat plate is given by $T = 100 - 0.09x^2 - 0.16y^2$, where x and y are the vertical and horizontal distances from a fixed point $(0, 0)$, measured in feet, and T is measured in degrees Fahrenheit. Consider the point $(5, 2)$.

   (a) In what direction must a bug move from $(5, 2)$ in order for temperature to decrease at the fastest rate? What is this rate (in degrees per foot)?

   (b) If the bug moves at 2ft/min in the above direction, how fast is the temperature felt by the bug decreasing (in degrees per minute)?

   (c) In what direction from $(5, 2)$ must the bug move so that the temperature neither increases nor decreases?

   (d) Another bug is moving along the curve of constant temperature, starting at $(5, 2)$ in the direction found in (c). It is moving at a constant speed of 3ft/min. Determine if the bug will return to $(5, 2)$, and if so, how long it will take.

5. Does the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} \exp\left(\int_0^{2n\pi} |\cos(nx)|\, dx\right)$ converge or diverge? If it converges, find its limit.

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6. Answer the following:

   (a) Define $\sum_{n=1}^{\infty} a_n$.

   (b) Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

   (c) Find the following limits and justify your answers rigorously. (Detailed $\varepsilon - \delta$ arguments are not necessary, but clearly explain the method you used.)

      (i) $\displaystyle\lim_{n\to\infty} \frac{1000^n}{n!}$

      (ii) $\displaystyle\lim_{n\to\infty} \left(1+\frac{k}{n}\right)^{n^2}$ (Hint: Consider the cases $k > 0$, $k < 0$ and $k = 0$ separately.)

      (iii) $\displaystyle\lim_{n\to\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3$

7. Answer the following:

   (a) What is the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} nx^n$?

   (b) What is the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n+1}$?

   (c) By differentiation or integration or some other process find the limits of the above series.

   (d) How many terms are necessary to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ within an error of 0.0001?

8. Consider the function $f(x) = \dfrac{x^2}{1+x^2}$ **with $\boldsymbol{x > 0}$**. Find the point on the graph of f for which the $x-$intercept of the tangent is largest.

9. A square $n \times n$ matrix A is called idempotent if $A^2 = A$.

   (a) Show: If λ is an eigenvalue of an idempotent matrix A, then $\lambda \in \{0, 1\}$.

   (b) Find an example of a 2×2 idempotent matrix whose entries are all nonzero.

   (c) Show that any idempotent matrix is diagonalizable.

10. Consider the matrix

$$B = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}.$$

    Here $a \in \mathbb{R}$ is a parameter.

    (a) For which values of a does B have rank 1?

    (b) Find an explicit formula for the nth power B^n.

END OF EXAM