

EXPLAIN ALL YOUR ANSWERS AND SHOW ALL YOUR WORK.
NO CALCULATORS. ALL ELECTRONIC DEVICES MUST BE TURNED OFF.

1. Let

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}.$$

- (i) Calculate the determinant of M .
- (ii) Find the eigenvalues and corresponding eigenvectors of M .
- (iii) Calculate

$$M^{2013} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. The probability that a random variable has a value between $-x$ and x , for $x \geq 0$, is

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^x e^{-t^2/2} dt.$$

- (i) Calculate the rate at which this probability changes with x (for $x > 0$).
- (ii) Is the rate of change ever $1/\sqrt{2\pi}$? If so, for which x ?

3. In a neighborhood clinic, the number of patients can be described as a function of the number of doctors, x , and the number of nurses, y , by $f(x, y) = 1000x^{0.6}y^{0.3}$. With upcoming budget cuts, the clinic must reduce the number of doctors at the rate of 2 per month. Find the rate at which the number of nurses has to be increased in order to maintain the current service (i.e., maintain the number of patients). Currently there are 30 doctors and 50 nurses.

4. Find the volume of the region in \mathbb{R}^3 with $x^2 + y^2 + z^2 \leq 2$ and $z^2 \leq x^2 + y^2$.

5. Let $M_{2 \times 2}$ be the vector space of all 2×2 matrices with real entries. Define $T : M_{2 \times 2} \rightarrow M_{2 \times 2}$ by

$$T(A) = A + A^T, \text{ where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- (i) Prove that T is a linear transformation.
- (ii) Let B be any element of $M_{2 \times 2}$ such that $B^T = B$. Find an A in $M_{2 \times 2}$ such that $T(A) = B$.
- (iii) Find the range of T .
- (iv) Find the kernel of T .

6. Define f on \mathbb{R}^2 by

$$f(x, y) = \begin{cases} 3, & \text{if } |x| + |y| \leq 2 \\ 5, & \text{if } |x| + |y| > 2. \end{cases}$$

Let $D = \{(x, y) \mid -3 \leq x \leq 3, -2 \leq y \leq 2\}$. Calculate the integral of f on D .

7. Find a point on the surface $x - yz = 3$ that is closest to the origin.

8. A 30-gallon tank initially contains 15 gallons of saltwater containing 6 pounds of salt. Suppose that saltwater containing 1 pound of salt per gallon is pumped into the top of the tank at the rate of 2 gallons per minute, while a well-mixed solution leaves the bottom of the tank at a rate of 1 gallon per minute. How much salt is in the tank when the tank is full?

9. By considering a Riemann sum, evaluate

$$\lim_{n \rightarrow \infty} \left(\left(\frac{1}{n} \sum_{k=1}^n \ln(n+k) \right) - \ln n \right).$$

10. If p is the probability that an experiment fails, where $0 < p < 1$, then the expected number of failures before a success is achieved is

$$E = (1-p) \sum_{k=1}^{\infty} kp^k.$$

Calculate E as a function of p that does not use a summation. (Hint: Consider $\sum kp^{k-1}$.)