Qualifying Exam

You may use calculators for this exam. Be advised, however, that every question can be answered without the use of a calculator and more than likely can be answered more efficiently without the use of a calculator. — Justify all your answers. Answer specific questions by giving the exact values, not approximations.

Problem 1. Consider the function

$$f(x) = \ln\left(\frac{1}{\sqrt{2 - e^{-2x}}}\right).$$

Determine the exact domain and range of this function. Prove that this function is an involution. Recall that a function $f : A \to A$ is an *involution* if f is a bijection and $f^{-1} = f$.

Solution. This function is defined whenever $e^{-2x} < 2$, and this is the case whenver $x > (-\ln 2)/2$. Thus, the domain of this function is the set $\{x \in \mathbb{R} : x > (-\ln 2)/2\} = ((-\ln 2)/2, +\infty)$. To determine the range, start with the range of $x \mapsto \sqrt{2 - e^{-2x}}$. The range of this function is $[0, \sqrt{2})$. Thus, the range of $x \mapsto \ln\sqrt{2 - e^{-2x}}$ is $(-\infty, \ln\sqrt{2})$. Hence, the range of the given function, $x \mapsto -\ln\sqrt{2 - e^{-2x}}$ is $(-\ln\sqrt{2}, +\infty)$, identical to its domain. The given function is a composition of bijections, so it is a bijection. To prove that f is an involution, notice that it can be written as $f(x) = (-1/2)\ln(2 - e^{-2x})$. With this expression, for $x > (\ln 2)/2$ we calculate
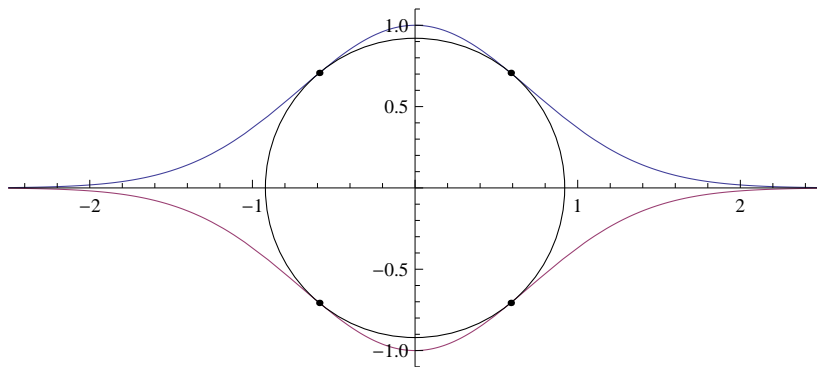
$$f(f(x)) = -\frac{1}{2}\ln\left(2 - e^{\ln(2 - e^{-2x})}\right) = -\frac{1}{2}\ln\left(2 - (2 - e^{-2x})\right) = -\frac{1}{2}\ln\left(e^{-2x}\right) = -\frac{1}{2}(-2x) = x.$$

Since at each step in the above simplification the expressions involved are defined, the simplification proves that f is an involution. □

Problem 2. The figure on the right shows the functions

$$y = e^{-x^2} \quad \text{and} \quad y = -e^{-x^2}$$

and the circle centered at the origin that touches both graphs. Find the exact value of the radius of this circle.

The phrase *"circle touches a graph"* means that the circle and the graph have a common point at which they have a common tangent line.

Solution. We seek the point $\left(x_0, e^{-x_0^2}\right)$ at which the normal to the graph $y = e^{-x^2}$ goes through the origin. The equation of a normal at this point is $y = (x - x_0)\left(e^{x_0^2}\right)/(2x_0) + e^{-x_0^2}$. To find that normal that goes through the origin we solve $-x_0\left(e^{x_0^2}\right)/(2x_0) + e^{-x_0^2} = 0$ for x_0, yielding $x_0 = -\sqrt{(\ln 2)/2}$ and $x_0 = \sqrt{(\ln 2)/2}$. Now, calculating the distance of the corresponding point on the graph to the origin gives us the radius: $\sqrt{(1 + \ln 2)/2} \approx 0.920094$. This result is consistent with what we see in the provided picture. □

Problem 3. Find all positive reals a for which the solution of the initial value problem

$$y' = \frac{1}{(1 + t^2)\, y}, \qquad y(0) = a$$

is defined for all $t \in \mathbb{R}$.

Solution. This is a separable differential equation. After separating variables we get $yy' = (1 + t^2)^{-1}$. The left hand side is the derivative of $y^2/2$. Therefore, $y^2/2 = C + \arctan t$. Now we can solve the initial value problem: $a^2/2 = C$. Thus, the solution of the initial value problem is $y^2 = a^2 + 2\arctan t$. Since the left hand side in the last equality is always nonnegative, for the last equation to be consistent for all $t \in \mathbb{R}$ we must have $a^2 + 2\arctan t \geq 0$ for all $t \in \mathbb{R}$. Since $\lim_{t \to -\infty} 2\arctan t = -\pi$, we must have $a^2 - \pi \geq 0$. Thus, the solution of the give initial problem is defined for all $t \in \mathbb{R}$ if and only if $a \geq \sqrt{\pi}$. If this condition is satisfied, the solution is given as $y(t) = \sqrt{a^2 + 2\arctan t}, t \in \mathbb{R}$. □

Problem 4. Consider the matrix $\quad M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$

(a) Write a basis for the column space and a basis for the null-space of M.

(b) Calculate all eigenvalues of M.

(c) Is M diagonalizable?

Solution. (a) Clearly the rank of M is 2. The first two columns of M form a basis for the column space of M. By the rank-nullity theorem the null-space of M is three dimensional. One can guess (or do a row reduction) a basis for the null space: $[0, 1, -1, 0, 0]^\top$, $[0, 1, 0, -1, 0]^\top$, $[0, 1, 0, 0, -1]^\top$.

(b) Denote the first column of M by $\mathbf{u}$ and the second one by $\mathbf{v}$. Then, $M\mathbf{u} = 4\mathbf{v}$ and $M\mathbf{v} = \mathbf{u}$. Therefore, the matrix representation of the restriction of M onto the column space with respect to the basis $\{\mathbf{u}, \mathbf{v}\}$ is the matrix

$$B = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}.$$

This matrix has eigenvalues 2 and -2 with a corresponding eigenvectors $[1\ 2]^\top$ and $[1\ -2]^\top$. Thus, M has eigenvalues 2 and -2 and the corresponding eigenvectors are $[2, 1, 1, 1, 1]^\top$ and $[-2, 1, 1, 1, 1]^\top$. In conclusion, M has three eigenvalues, $-2, 0, 2$. The eigenspaces of -2 and 2 are one-dimensional, while the eigenspace of 0 is three dimensional.

(c) Since B is invertible none of the nonzero vectors from the null-space of M belongs to the column space. Therefore, the five eigenvectors just found are linearly independent. Thus M is diagonalizable. In fact, setting

$$P = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we have $MP = PD$ which is equivalent to $P^{-1}MP = D$. $\quad\square$

Problem 5. Let $A = \begin{bmatrix} 5/2 & -1 \\ 3 & -1 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Calculate $\lim_{k\to\infty} A^k \mathbf{x}_0$.

Solution. A simple calculation shows that the eigenvalues of A are 1 with a corresponding eigenvector $[2\ 3]^\top$ and $1/2$ with a corresponding eigenvector $[1\ 2]^\top$. Next we represent the given vector $\mathbf{x}_0$ as the linear combination of eigenvectors: $[1\ 1]^\top = [2\ 3]^\top - [1\ 2]^\top$. Thus, $A^k \mathbf{x}_0 = [2\ 3]^\top - (1/2)^k [1\ 2]^\top$. Therefore, $\lim_{k\to\infty} A^k \mathbf{x}_0 = [2\ 3]^\top$. $\quad\square$

Problem 6. Let $\{a_n\}$ be a sequence of positive real numbers.

(a) Prove or disprove: If $\sum_{n=1}^\infty a_n$ converges, then $\sum_{n=1}^\infty \sqrt{a_n a_{n+1}}$ converges.

(b) Prove or disprove: If $\sum_{n=1}^\infty \sqrt{a_n a_{n+1}}$ converges, then $\sum_{n=1}^\infty a_n$ converges.

(c) In addition assume that $\{a_n\}$ is nonincreasing. With this additional assumption, prove or disprove the above implications.

Hint: The inequality of arithmetic and geometric means can be very useful here. Write this inequality down before proceeding.

Solution. As hinted, let us state the inequality of arithmetic and geometric means. If x and y are nonnegative real numbers, then, $\sqrt{xy} \le (x + y)/2$.

(a) is true. Assume that $\sum_{n=1}^{\infty} a_n$ converges. Then $\sum_{n=1}^{\infty} a_{n+1}$ converges and also $\sum_{n=1}^{\infty}(a_n+a_{n+1})/2$ converges. By AM-GM inequality for every $n \in \mathbb{N}$ we have $\sqrt{a_n a_{n+1}} \leq (a_n+a_{n+1})/2$. Now by the comparison test $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges.

(b) is false. An example showing this is as follows. Let $a_{2k} = 1/k$ and $a_{2k+1} = 1/(k^3)$. Then $\sqrt{a_{2k}a_{2k+1}} = 1/(k^2)$ and $\sqrt{a_{2k+1}a_{2k+2}} = 1/\sqrt{k^3(k+1)} < 1/(k^2)$. Thus $\sum \sqrt{a_n a_{n+1}}$ converges by the comparison test and $\sum a_n$ diverges.

(c) With this additional assumption, (b) is true. The additional assumption implies that $a_{n+1} \leq \sqrt{a_n a_{n+1}}$, so by the comparison test $\sum a_{n+1}$ converges whenever $\sum \sqrt{a_n a_{n+1}}$ converges. Since $\sum a_{n+1}$ converges if and only if $\sum a_n$ converges, (b) is proved. $\qquad\square$

Problem 7. Find the volume of the solid given by

$$W = \{(x,y,z) \in \mathbb{R}^3 : x^2 + z^2 \leq 1, \ y^2 + z^2 \leq 1\}$$

by slicing the solid by planes parallel to the xy-plane. See Figure 1.

Solution. A horizontal plane at level z, $-1 \leq z \leq 1$ intersects two cylinders at a square with sides $2\sqrt{1 - z^2}$. Thus the volume that we are interested in is given by the integral

$$4 \int_{-1}^{1} (1 - z^2)dz = \frac{16}{3}.$$

$\square$

Remark. This is well known example of finding volumes by slicing. This problem is in many calculus books. The animation tittled: "a cross-section is a square" posted on Wednesday, February 25, 2009 at this website is relevant to this problem.
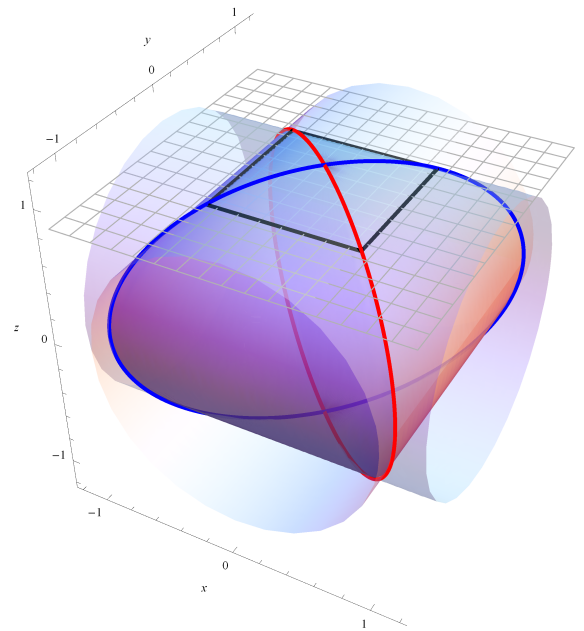

Figure 1: Two cylinders

Problem 8. Figure 3 shows a right circular cone and a sphere completely contained in the cone. Find the radius of the base and the height of the right circular cone with the smallest surface area which completely contains the unit sphere. The mesh that you see in the picture is not part of the surface area of the cone; it is there just to indicate that the sphere is completely contained in the cone.
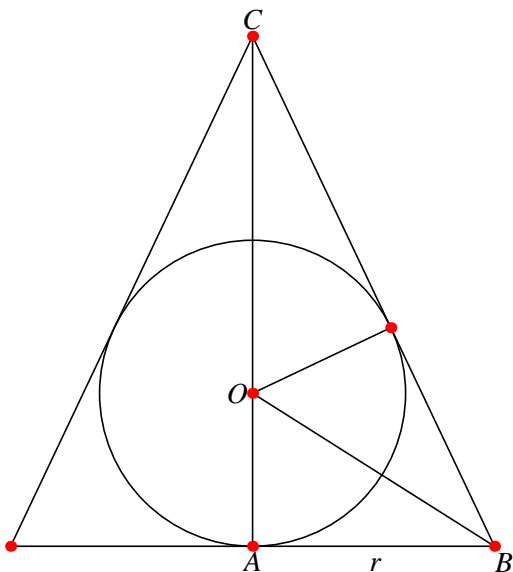
Figure 2: Triangle ABC
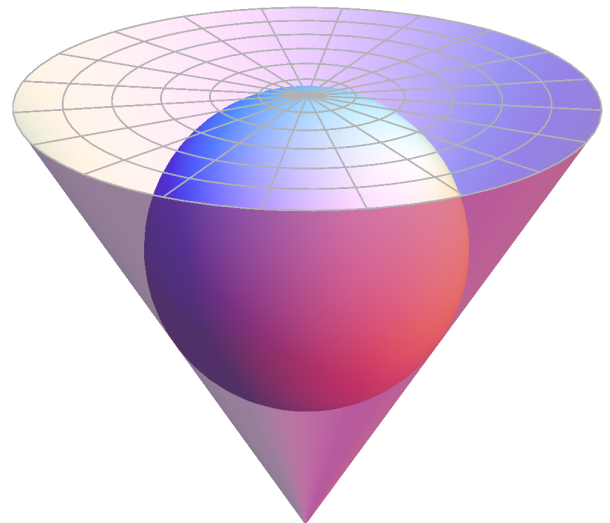
Figure 3: Sphere in a cone

Solution. First we have to determine which isosceles triangles contain the unit circle. Figure 2 shows an isosceles triangle with its inscribed circle. We assume that the radius of the inscribed circle is 1. Denote by r the lenght of the line segment AB. Denote the length of AC by h and the length of BC by s. Notice that h is the height of the cone and s is a directrix of the cone. We need to calculate h and s in terms of r. Further notice that $1/r$ is the tangent of the angle $\angle ABO$. We use the double-angle formula $\tan(2\alpha) = 2(\tan\alpha)/(1 - (\tan\alpha)^2)$ and the fact that the angle $\angle ABC$ is twice $\angle ABO$ to calculate the tangent of $\angle ABC$ to be $(2/r)/(1 - 1/r^2) = 2r/(r^2 - 1)$. Since the tangent of $\angle ABC$ is h/r, we get that $h = 2r^2/(r^2 - 1)$. From the Pythagorean theorem we get

$$s = \sqrt{r^2 + \frac{4r^4}{(r^2 - 1)^2}} = \frac{r(r^2 + 1)}{r^2 - 1}$$

The surface area of the cone with the directrix s and radius of the base r is given by $sr\pi$. Thus we need to minimize

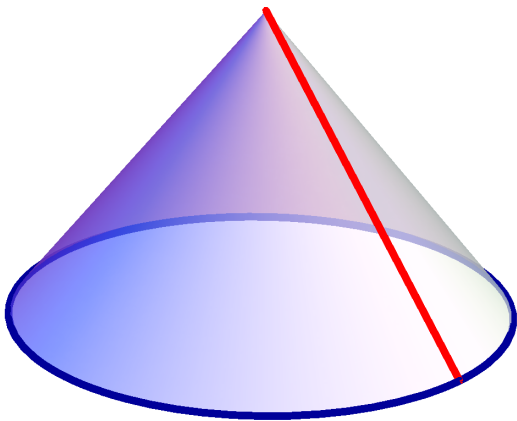$$sr = \frac{r^2(r^2 + 1)}{r^2 - 1}, \qquad r > 1.$$

The derivative with respect to r of the above function is

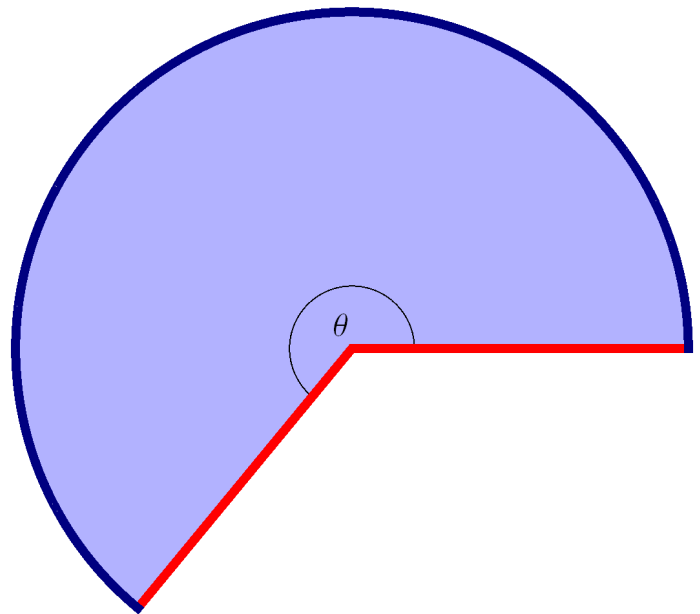$$\frac{2r}{(r^2 - 1)^2}\left(r^4 - 2r^2 - 1\right).$$

Since $2r/((r^2 - 1)^2) > 0$ for $r > 1$ and

$$r^4 - 2r^2 - 1 = \left(r^2 - 1 + \sqrt{2}\right)\left(r^2 - 1 - \sqrt{2}\right)$$

we conclude that the minimum area occurs at $r = \sqrt{1 + \sqrt{2}}$. The corresponding $h = 2 + \sqrt{2}$ and $s = \sqrt{7 + 5\sqrt{2}}$. $\square$

Figure 4: A cone with a red directrix and dark blue circumference of the base

Figure 5: The surface area of a cone

Remark. You are not expected to know the formula for the surface area of the cone. However, you are expected to be able to deduce this formula. When unwrapped the surface area of a cone is a circular sector, see Figures 4 and 5. This circular sector is cut out of a circle whose radius is s, where s is the length of a directrix of the cone shown in red in above figures. The length of the circular arc bounding this circular sector is $2r\pi$, where r is the radius of the base of the cone. This arc is shown in dark blue. **You are expected to know** the relationship between θ in Figure 5 and the lengths s and $2r\pi$ (that is the relationship between the radius, the arc length and θ: $\theta*(\text{red radius}) = \text{arc length}$): $\theta s = 2r\pi$. This determines $\theta = 2r\pi/s$. **You are expected to know** how to calculate the area of a circular sector when the radius and θ are known ($\text{area} = (\text{radius})^2*\theta/2$). Since in our case the radius is s, $\theta = 2r\pi/s$, we conclude that the area of the circular sector which represents the surface area of the cone is $sr\pi$.

Problem 9. Recall that $[0,1]^3$ denotes the unit cube in $\mathbb{R}^3$, see Figure 6. That is the set

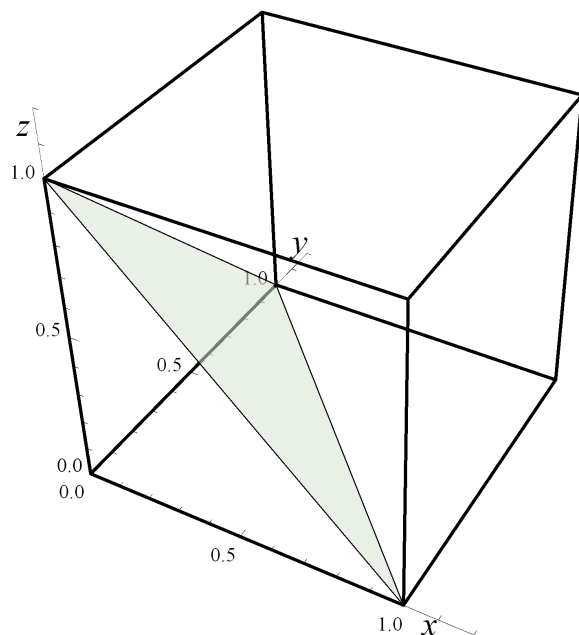$$[0,1]^3 = \left\{(x,y,z) \in \mathbb{R}^3 : x,y,z \in [0,1]\right\}.$$

Consider the function

$$f(x,y,z) = -\left(x \ln x + y \ln y + z \ln z\right), \quad (x,y,z) \in [0,1]^3.$$

Notice that $\lim_{t \to 0+} t \ln t = 0$. Therefore the function f is defined on all the sides of the unit cube.

(a) Determine the global maximum and the global minimum of the function f on the unit cube.

(b) Determine the global maximum and the global minimum of the function f on the intersection of the unit cube and the plane $x + y + z = 1$.

Figure 6: The unit cube

Solution. (a) We first explore the interior of the unit cube. We look at the points where the gradient of f equals the zero vector. That is where, $-\ln x - 1 = 0$, $-\ln y - 1 = 0$, and $-\ln z + 1 = 0$. Thus, the point $(1/e, 1/e, 1/e)$ is the only critical point of f and it is in the unit cube. The value of f at this point is $3/e$. Next we explore the sides of the unit cube. The identities $f(x,y,0) = f(x,y,1)$, $f(x,y,z) = f(y,z,x) = f(z,x,y) = f(z,y,x) = f(y,x,z) = f(x,z,y)$ imply that it is enough to consider one side of the cube. We consider the unit square in xy-plane, $f(x,y,0) = -\left(x \ln x + y \ln y\right)$. The maximum in the interior of this square is at $(1/e, 1/e, 0)$ and equals $2/e$. The maximum at the edges is $1/e$. The minimum is clearly 0 attained at each of the vertices. Hence the global maximum is $3/e$ attained at $(1/e, 1/e, 1/e)$ and the global minimum is 0 attained at any of the vertices.
(b) For an extreme value in the interior of the triangle, we are looking for the point in the triangle where the gradient is parallel to the normal vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$ of the plane. This leads to the equalities $-1 - \ln x = \lambda$, $-1 - \ln y = \lambda$, $-1 - \ln z = \lambda$, which imply that $x = y = z$. Together with $x + y + z = 1$ we obtain that the local extreme in the interior of the triangle is at the point $(1/3, 1/3, 1/3)$ with the value of f being $\ln 3$. Again, because of the symmetry it is enough to consider one edge: $f(x, 1 - x, 0) = -x \ln x - (1 - x) \ln(1 - x)$. This is a continuous nonngative function on $[0,1]$ which is "even" around the line $x = 1/2$. Thus its maximum is $\ln 2$ attained at $x = 1/2$. In conclusion, the global maximum is $\ln 3$ attained at $(1/3, 1/3, 1/3)$ and the global minimum is 0 attained at the vertices.
Notice that the global maximum $3/e \approx 1.10364$ and the constrained maximum $\ln 3 \approx 1.09861$ are very close. This solution is a very round about proof that $3 > e \ln 3$. A more direct proof would be to consider the function $t \mapsto t - e \ln t$, $t > 0$, and prove that this function is nonnegative with the global minimum only at $t = e$. $\qquad \square$

Problem 10. Let $r > 0$. Let W_r be the part of the unit sphere centered at the origin which is cut out by the cone $z = r\sqrt{x^2 + y^2}$. That is

$$W_r = \{(x, y, z) \; : \; x^2 + y^2 + z^2 \leq 1, \; r\sqrt{x^2 + y^2} \leq z\}.$$

Calculate r for which the volume of W_r equals one third of the volume of the unit sphere. See Figure 7 below.
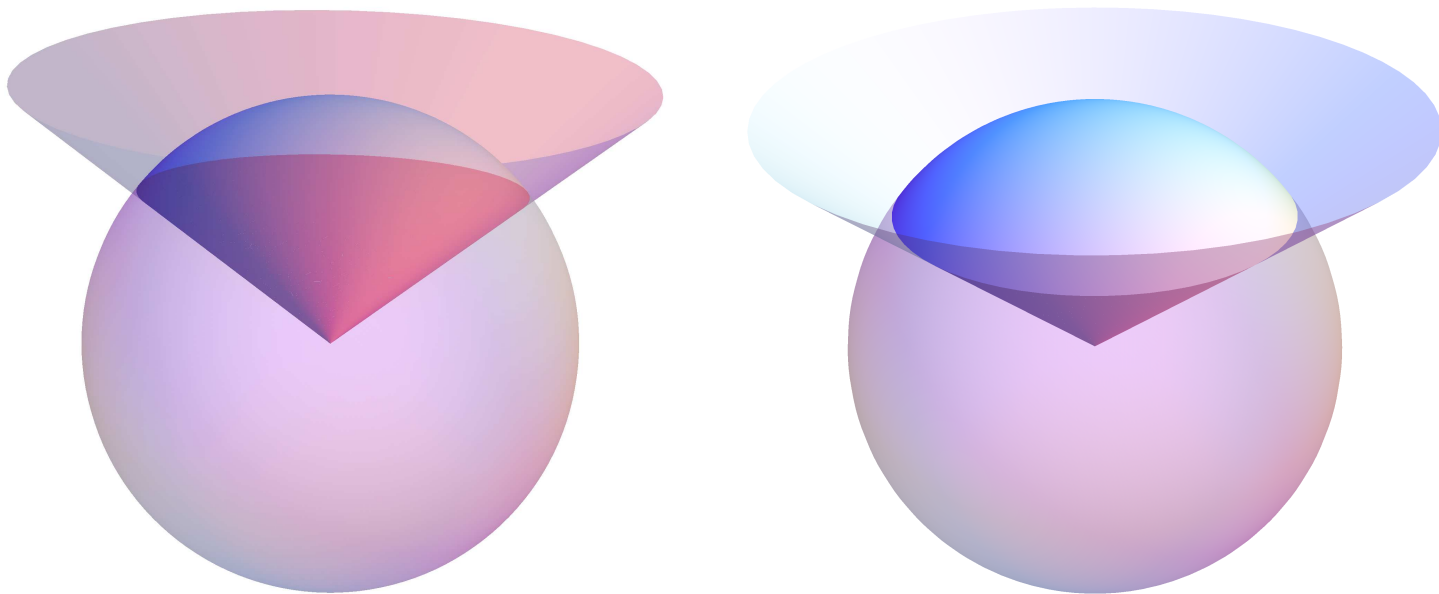
Figure 7: Two views of the region W_r

Solution. The volume of the regin W_r in spherical coordinates is given by

$$\int_0^{2\pi} \int_0^1 \int_0^{\arctan(1/r)} \rho^2 \sin\phi \; d\phi \, d\rho \, d\theta = \frac{2\pi}{3}\big(1 - \cos(\arctan(1/r))\big).$$

To get $1/3$ of a sphere we solve for r

$$\frac{2\pi}{3}\big(1 - \cos(\arctan(1/r))\big) = \frac{1}{3}\frac{4\pi}{3},$$

that is $\cos(\arctan(1/r)) = \frac{1}{3}$, which leads to

$$\frac{1}{r} = \tan\big(\arccos(1/3)\big) = \big(\sin(\arccos(1/3))\big)/\cos(\arccos(1/3)) = \big(\sqrt{1 - 1/9}\big)/(1/3) = 2\sqrt{2},$$

that is $r = 1/(2\sqrt{2})$.   □