Qualifying Exam: Fall 2015

1. The length of the curve which is the graph of a function $y = f(x)$ between $x = a$ and $x = b$ is equal to $\int_a^b \sqrt{1 + f'(x)^2}\, dx$. Consider the curve given by the graph of $y = f(x) = 2\sqrt{x}$. Let $a(t) = t$ and $b(t) = 2t + 1$. Let $L(t)$ be the length of the curve given by the graph of $f(x)$ between $x = a(t)$ and $x = b(t)$.

   Find t^*, the value of $t > 0$ where the length $L(t)$ is **minimal**. Prove that it is indeed a minimum.

   Ans. We have $L(t) = \int_t^{2t+1} \sqrt{1 + \frac{1}{x}}\, dx$, so by the fundamental theorem of calculus,

   $$L'(t) = \sqrt{1 + \frac{1}{2t+1}} \times 2 - \sqrt{1 + \frac{1}{t}}.$$

   Finding where $L'(t) = 0$,

   $$4 + \frac{4}{2t+1} = 1 + \frac{1}{t} \qquad \Rightarrow \qquad 6t^2 + 5t - 1 = 0.$$

   The positive root of this is $t^* = \frac{1}{6}$. To see it is indeed a minimum,

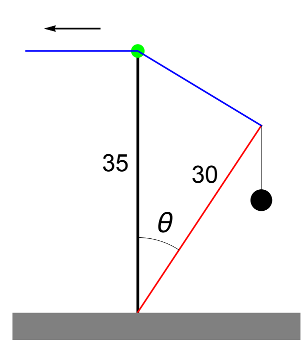
   $$L''(t) = \frac{2}{2\sqrt{1 + \frac{1}{2t+1}}} \frac{-2}{(2t+1)^2} - \frac{1}{2\sqrt{1 + \frac{1}{t}}} \frac{-1}{t^2}$$

   $$L''(t^*) = -\frac{2}{\sqrt{7/4}} \frac{9}{16} + \frac{36}{2\sqrt{7}} = \frac{9\sqrt{7}}{4} > 0$$

   which shows it is a minimum.

2. The device shown in the picture is called a "derrick" – the vertical $35m$ pole (black) is fixed and has a pulley at the top (green). A cable (blue) runs over the pulley and is attached to the end of the $30m$ boom (red) which is free to rotate about its lower end. There is a weight hanging from the end of the boom on a cable of fixed length.

   If the (blue) cable is drawn at the rate of $2m/s$ to lift the boom, how quickly is the weight being raised *vertically* at the instant that θ, the angle between the pole and the boom, is $12°$?

   In case you've forgotten,

   $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$
   $$a^2 = b^2 + c^2 - 2BC\cos A,$$
   $$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B,$$
   $$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

Ans. Let $h(t)$ be the height of the weight. Then $\cos\theta = \dfrac{h(t)}{30}$ and so $-\sin\theta\dfrac{d\theta}{dt} = \dfrac{1}{30}\dfrac{dh}{dt}$, or $\dfrac{dh}{dt} = -30\sin\theta\dfrac{d\theta}{dt}$.

Let a be the length of the side opposite θ; we know that $\dfrac{da}{dt} = -2$. By the law of cosines, $\cos\theta = \dfrac{1}{2\times 30\times 35}(30^2+35^2-a^2)$, so $\sin\theta\dfrac{d\theta}{dt} = \dfrac{2a}{2\times 30\times 35}\dfrac{da}{dt} = \dfrac{-2a}{30\times 35}$. Also by the law of cosines, when $\theta = 12^\circ$, $a^2 = 30^2+35^2-2\times 30\times 35\cos\dfrac{12\pi}{180}$ from which we find $a = 8.41962$.

Combining these, $\dfrac{dh}{dt} = -30\sin\theta\dfrac{d\theta}{dt} = -30\times\dfrac{-2a}{30\times 35} = \dfrac{2\times 8.41962}{35} = 0.4811$.

3. (a) Suppose that the MacLaurin series (Taylor series based at 0) for the function

$$f(x) = \frac{1+x}{1+ax} - \ln(1+bx)$$

is such that the coefficients of x and x^2 are both zero. Find two possible pairs of the values of the constants a and b. Verify that for one of these pairs of values, in fact, all coefficients of powers of x in the series expansion of $f(x)$ are zero.

(b) Prove or disprove: $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}}\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\Big) = 0$.

Ans. (a) Computing $f'(0) = 1-a-b$ and $\dfrac{1}{2}f''(0) = a^2-a+\dfrac{b^2}{2}$ and solving these to simultaneously equal 0 we obtain

$$b = 1-a \quad\Rightarrow\quad 3a^2-4a+1 = 0 \quad\Rightarrow\quad a = 1 \text{ or } a = \frac{1}{3}.$$

Thus the two choices are $a = 1,\ b = 0$, and $a = \dfrac{1}{3},\ b = \dfrac{2}{3}$.

(b) Comparing to an integral, we see that $\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n} < \displaystyle\int_1^n \frac{1}{x}\,dx = \ln(n)$. Thus

$$\frac{1}{\sqrt{n}}\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\Big) < \frac{1}{\sqrt{n}}(1+\ln(n)) \to 0$$

as $n\to\infty$ by l'Hopital's rule.

4. (a) Prove that $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges if $\displaystyle\lim_{n\to\infty} na_n = A > 0$. Hint: Think harmonic series.

(b) Using part (a), or some other method, show that $\displaystyle\sum_{n=1}^{\infty}\ln\big(1+\frac{2}{n}\big)$ diverges.

(c) Determine whether or not $\displaystyle\sum_{n=1}^{\infty}\frac{n^{2n}}{7^n(n!)^2}$ converges.

Ans. (a) Set $\varepsilon = A/2 > 0$. For sufficiently large N, $A/2 < na^n < 3A/2$ for all $n > N$. Thus, for all $n > N$, $a_n > \dfrac{A}{2n}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so too does $\sum a_n$, by the comparison test.

(b) $\displaystyle\lim_{n\to\infty} n \ln\bigl(1 + \frac{2}{n}\bigr) = 1$ by l'Hopital's rule. Thus the original series diverges by (a).

(c) Computing,

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^{2n+2}7^n(n!)^2}{7^{n+1}((n+1)!)^2 n^{2n}} = \frac{1}{7}\Bigl(\frac{n+1}{n}\Bigr)^{2n}\frac{(n+1)^2 n!n!}{(n+1)!(n+1)!}\\ &= \frac{1}{7}\Bigl[\bigl(1+\frac{1}{n}\bigr)^n\Bigr]^2 \to \frac{e^2}{7} > 1,\end{aligned}$$

so the series diverges by the Ratio test.

5. Let A be an $n \times n$ real matrix.

   (a) Show that if $A^T = -A$ and n is odd, then $\det A = 0$.

   (b) Let $C = BA$ for some nonsingular matrix B. Show that rank$(C) =$ rank(A).

   (c) Let A be an **orthogonal** matrix. Show that Range$(I - A) \subseteq \big(\text{Null}(I - A)\big)^{\perp}$, that is, every element in the range (image) of $I - A$ is perpendicular to every element in the null space (kernel) of $I - A$.

   Ans. (a) Since $\det A^T = \det A$ we have $\det A = \det(-A) = (-1)^n \det A = -\det A$ if n is odd. Thus $\det A = 0$.

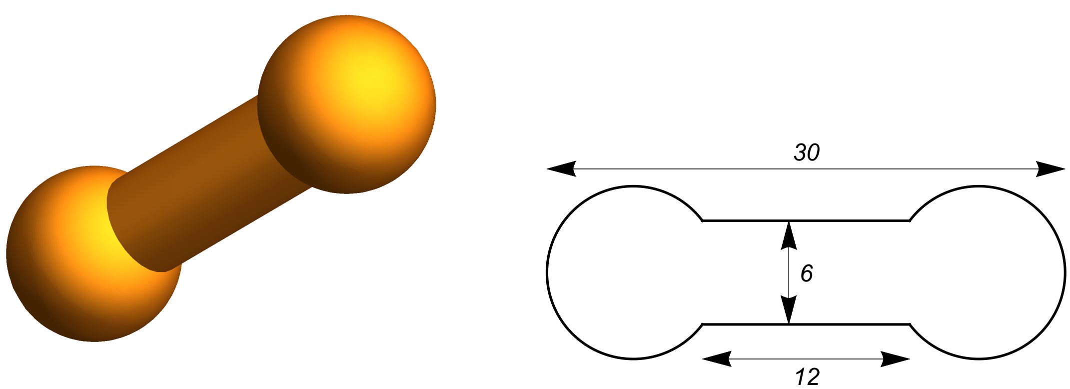
   (b) If $\vec{y} \in$ Range(C) then $\vec{y} = C\vec{x} = AB\vec{x} = A(B\vec{x})$ for some $\vec{x}$ and so $\vec{y} \in$ Range(A). Conversely, if $\vec{y} \in$ Range(A) then $\vec{y} = A\vec{x}$ form some vector $\vec{x}$. Set $\vec{z} = B^{-1}\vec{x}$ (since B is nonsingular); then $\vec{y} = AB\vec{z} = C\vec{z}$ and so $\vec{y} \in$ Range(C). Thus Range$(C) =$ Range(A) and so rank$(C) =$ rank(A).

   (c) Let $\vec{y} \in$ Range$(I - A)$, so $\vec{y} = (I - A)\vec{x}$ for some $\vec{x}$. Now let $\vec{u} \in$ Null$(I - A)$ so $(I - A)\vec{u} = \vec{0}$. We compute

$$\vec{y}^T\vec{u} = \vec{x}^T(I - A^T)\vec{u} = \vec{x}^T(A^TA - A^T)\vec{u} = \vec{x}^TA^T(A - I)\vec{u} = 0$$

   and so Range$(I - A) \subset \big(\text{Null}(I - A)\big)^{\perp}$. To prove the reverse containment, $\big(\text{Null}(I - A)\big)^{\perp} \subset$ Range$(I - A)$ it is easier, and equivalent, to prove that Null$(I - A) \subset \big(\text{Range}(I - A)\big)^{\perp}$. For this, let $\vec{u} \in$ Null$(I - A)$, $(I - A)\vec{u} = \vec{0}$, and let $\vec{y} = (I - A)\vec{x} \in$ Range$(I - A)$. Computation of $\vec{y}^T\vec{u} = 0$ is the same as before.

6. Shown is a dog's toy made from two (partial) balls attached onto a (solid) cylindrical rod; the dimensions (in cm) are given in the cross-section. Compute the volume of the toy.

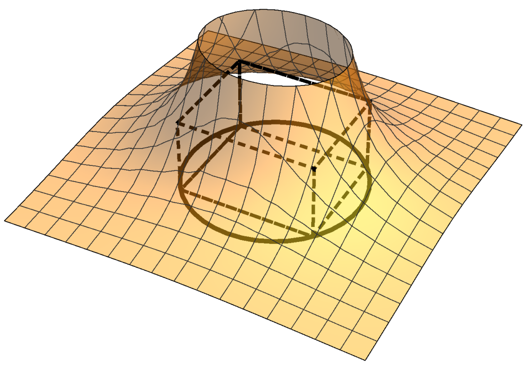


   Ans. The volume of the cylinder is clearly $\pi \times 3^2 \times 12 = 108\pi$. To compute the volume of one of the partial balls, we center the ball at the origin and consider $-R < z < z_0$; we must determine z_0 and R, where R is the radius of the ball. Drawing the appropriate triangle we find $3^2 + z_0^2 = R^2$ and $2(z_0 + R) + 12 = 30$. This second implies $R^2 = 81 - 18z_0 + z_0^2$ from which we derive $z_0 = 4$ and then $R = 5$. Using cylindrical coordinates, the volume of the partial ball is thus

$$2\pi \int_{-5}^{4} \int_{0}^{\sqrt{25-z^2}} r\, dr\, dz = 2\pi \int_{-5}^{4} \frac{1}{2}(25 - z^2)\, dz = 2\pi\Big(\frac{225}{2} - \frac{189}{6}\Big) = 162\pi.$$

The total volume is thus $2 \times 162\pi + 108\pi = 432\pi$.

7. Consider a rectangular box whose bottom corners lie on the unit circle centered at the origin in the xy-plane, and whose top corners meet the surface $f(x,y) = \dfrac{1}{2x^2+3y^2}$, and whose sides are parallel to the coordinate axes. One possible such rectangular box is shown in the figure. Find the **volume** of the box which is **maximal** among all such boxes.



Ans. Let $V(x,y) = \dfrac{4xy}{2x^2+3y^2}$, and $g(x,y) = x^2+y^2-1$. Then

$$\nabla V(x,y) = \lambda \nabla g(x,y) \qquad \Rightarrow \qquad \Big(\frac{4y(3y^2-2x^2)}{(2x^2+3y^2)^2}, \frac{-4y(3y^2-2x^2)}{(2x^2+3y^2)^2}\Big) = \lambda(2x, 2y).$$

From this we obtain (for $x \neq 0$ and $y \neq 0$, which cannot be maximal)

$$\frac{4y}{x}(3y^2-2x^2) = -\frac{4x}{y}(3y^2-2x^2) \quad \Rightarrow \quad y^2 = -x^2 \text{ or } 3y^2 = 2x^2.$$

The first is impossible; the second, together with the constraint, gives $3y^2 = 2(1-y^2) \Rightarrow y = \pm\sqrt{\frac{2}{5}}$, and then $x = \pm\sqrt{\frac{3}{5}}$. The maximal volume is thus

$$V\Big(\sqrt{\frac{3}{5}}, \sqrt{\frac{2}{5}}\Big) = \frac{4\frac{\sqrt{6}}{5}}{\frac{6}{5}+\frac{6}{5}} = \frac{\sqrt{6}}{3} = \sqrt{\frac{2}{3}}.$$

8. Let $\mathcal{A}$ be a linear transformation, $\mathcal{A} : V \to V$, where V is an n-dimensional vector space. Suppose that $\vec{x} \in V$ is a non-zero vector such that $\mathcal{A}^{n-1}\vec{x} \neq \vec{0}$, but $\mathcal{A}^n\vec{x} = \vec{0}$.

   (a) Show that $\{\vec{x}, \mathcal{A}\vec{x}, \mathcal{A}^2\vec{x}, \ldots, \mathcal{A}^{n-1}\vec{x}\}$ is a linearly independent set.

   (b) Recall that an eigenvalue of a linear transformation T is a scalar λ for which the equation $T\vec{v} = \lambda\vec{v}$ has a non-zero solution $\vec{v}$. What are the eigenvalues of $\mathcal{A}$? Hint: Note that the set in part (a) forms a basis for V.

Ans. (a) Suppose that $c_0\vec{x} + c_1\mathcal{A}\vec{x} + \cdots + c_{n-1}\mathcal{A}\vec{x} = \vec{0}$. Applying $\mathcal{A}^{n-1}$ to both sides yields $c_0\mathcal{A}^{n-1}\vec{x} = \vec{0}$, and since $\mathcal{A}^{n-1}\vec{x} \neq \vec{0}$, we conclude $c_0 = 0$. Subsequently applying $\mathcal{A}^{n-2}$, $\mathcal{A}^{n-3}$,...,$\mathcal{A}$ we conclude $c_j = 0$ for all $j = 0, \ldots, n-1$ and so the set is linearly independent.

(b) Suppose that $\vec{u} \neq \vec{0}$ is an eigenvector for the eigenvalue λ, so $\mathcal{A}\vec{u} = \lambda\vec{u}$. Writing

$$\vec{u} = r_0\vec{x} + r_1\mathcal{A}\vec{x} + \cdots + r_{n-1}\mathcal{A}^{n-1}\vec{x},$$

we have

$$r_0\mathcal{A}\vec{x} + r_1\mathcal{A}^2\vec{x} + \cdots + r_{n-2}\mathcal{A}^{n-1}\vec{x} = \lambda(r_0\vec{x} + r_1\mathcal{A}\vec{x} + \cdots + r_{n-1}\mathcal{A}^{n-1}\vec{x}).$$

Since we have a basis, this implies that

$$0 = \lambda r_0, \quad r_0 = \lambda r_1, \quad r_1 = \lambda r_2, \quad \ldots, r_{n-2} = \lambda r_{n-1}.$$

If $\lambda \neq 0$ then $r_0 = 0 \Rightarrow r_1 = 0 \Rightarrow \cdots \Rightarrow r_{n-1} = 0$ in which case $\vec{u} = \vec{0}$, which contradicts $\vec{u}$ being an eigenvector. Thus $\lambda = 0$.

9. Consider the first order ordinary differential equation (ODE)

$$\frac{dy}{dx} + x|x|y = x|x|, \quad y(0) = y_0.$$

The theory of ODEs guarantees the existence and uniqueness of the solution for any initial condition y_0. You may take it as a fact that, for all integers $n > 1$, $x^n|x|$ is differentiable and $\frac{d}{dx}x^n|x| = (n+1)x^{n-1}|x|$.

   (a) Without solving the ODE, show that if $y(x)$ is a solution, then $x = 0$ is a local maximum for $y_0 > 1$ and a local minimum for $y_0 < 1$. (Hint: use the ODE to compute $y'(0)$, and to determine the sign of $y'(x)$ for x near $x = 0$.)

   (b) Solve the initial value problem. Note: it does not suffice to use your solution to (b) to answer (a) directly.

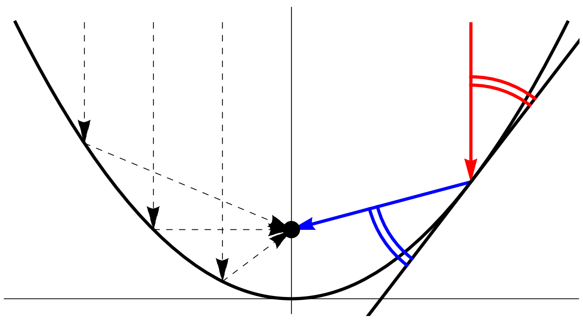
Ans. (a) First the ODE gives $y'(x) = x|x|(1 - y(x))$, so $y'(0) = 0$ and $x = 0$ is a critical point. If $y(0) > 1$ then (by continuity) $1 - y(x) < 0$ in a neighborhood of 0. Since $x|x|$ has sign chart $-; 0; +$ near 0, the product $x|x|(1 - y(x))$ has sign chart $+; 0; -$ which shows there is a local maximum at $x = 0$. Similarly, for $y(0) < 1$, the sign chart for $y'(x) = x|x|(1 - y(x))$ is $-; 0; +$ and there is a local minimum at $x = 0$.

(b) The ODE is linear; an integrating factor is $\exp(\int x|x|\,dx) = \exp\big(\frac{1}{3}x^2|x|\big)$, so the solution satisfies

$$\big(e^{x^2|x|/3}y\big)' = x|x|e^{x^2|x|/3} \quad\Rightarrow\quad e^{x^2|x|/3}y = \int x|x|e^{x^2|x|/3}\,dx + C = e^{x^2|x|/3} + C$$
$$\Rightarrow\quad y = 1 + Ce^{-x^2|x|/3} = 1 + (y_0 - 1)e^{-x^2|x|/3}.$$

10. Every parabola has a *focal point*: consider a ray parallel to the axis of the parabola meeting the parabola (red ray), then reflecting (blue ray) so that the angle between the reflected ray and the tangent line at the point of intersection (blue angle) is equal to the angle between the incoming ray and the same tangent line (red angle). See the picture. The focal point is on the axis of the parabola, and all such reflected rays meet at this point.

Let the parabola be $y = x^2$; let the focal point be $(0, \varphi)$. Find φ. Your derivation must also prove that φ is correct. (Hint: use vector calculus and the dot product.)



Ans. The parobola can be seen as the level-0 set of $f(x, y) = y - x^2$. The gradient vector at (x, x^2) is thus $(-2x, 1)$ and is perpendicular to the curve there. We seek a vector $\vec{v}$ so that $\vec{v} \cdot (-2x, 1) = (0, 1) \cdot (-2x, 1) = 1$, and $\|\vec{v}\| = 1$. Thus

$$-2xv_1 + v_2 = 1 \qquad \text{and} \qquad v_1^2 + v_2^2 = 1.$$

From these we obtain $v_1^2 + (1 + 2xv_1)^2 = 1$ and solving for v_1 we find $v_1 = 0$ (not the solution we want) and $v_1 = \dfrac{-4x}{1+4x^2}$. Then

$$v_2 = 1 + 2xv_1 = \frac{1 - 4x^2}{1 + 4x^2}.$$

Now the line from (x, x^2) in the direction v is then

$$(x, x^2) + t\Big(\frac{-4x}{1+4x^2}, \frac{1-4x^2}{1+4x^2}\Big) = (0, \varphi)$$

when $t = \dfrac{1+4x^2}{4}$. At this t,

$$\varphi = x^2 + \frac{1+4x^2}{4}\frac{1-4x^2}{1+4x^2} = \frac{1}{4}.$$