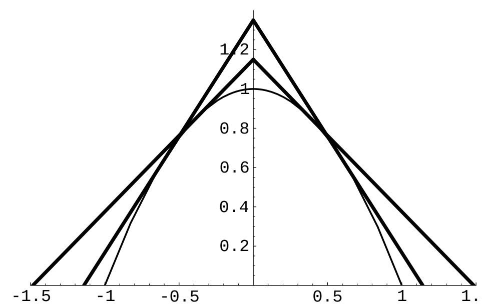
WWU Graduate Qualifying Exam, Spring 2008

You may use calculators for this exam. Be advised however that every question can be answered without the use of a calculator and more than likely can be answered more efficiently without the use of a calculator.

1. Define $f(t) = \displaystyle\int_1^{t^2} \frac{1}{s} e^{s^2 t}\, ds$. Find $\dfrac{df}{dt}$.

2. Consider the parabola $y = 1 - x^2$ and the triangle made from $y = a - bx$ (and its reflection $y = a + bx$) which has the property that the triangle is *tangent* to the parabola at the points of contact. The constants a and b are positive. Two examples are shown in the diagram.



   Find the numbers a and b which *minimize* the area under the triangle and above the x-axis (note, you can work with just one side of the symmetric problem).

3. (a) Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sinh n}{n^n}$ converges, and justify your answer.

   (b) Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \tan^{-1} n$ converges, and justify your answer.

   (c) Determine for what real values of x the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{2^{(n+1)}}$ converges and for what values it diverges. Make sure that every value of x is considered. Justify your answers.

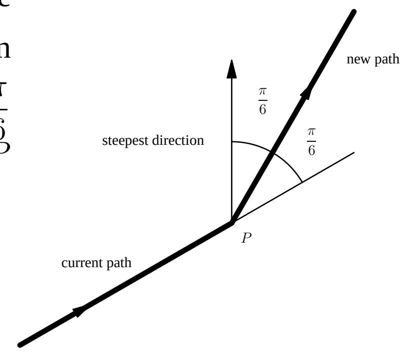
4. Consider the differential equation $\dfrac{dy}{dt} = -\dfrac{1}{t} y + t^{\alpha}$, valid for $t \geq 1$, and where $\alpha \in \mathbb{R}$ is a parameter.

   (a) Find the general solution to this ODE. Warning: pay attention to α.

   (b) Consider the long-time behavior ($t \to \infty$) of your solution(s): show that there is a value α^* such that the behavior for solutions for $\alpha < \alpha^*$ is different from the behavior of solutions for $\alpha > \alpha^*$. Describe these behaviors, and also for the case $\alpha = \alpha^*$.

   (c) Solve the initial value problem $y(1) = 2$ for the case $\alpha = 2$.

5. While hiking along a trail the elevation increases at a rate of $\frac{1}{3}$ meters per meter. At a point P the "current" path then veers more up-hill to a "new" path, making an angle of $\frac{\pi}{6}$ with the current path. The steepest direction up-hill from P makes an angle of $\frac{\pi}{3}$ with the current path.



   (a) At what rate will the elevation be increasing when you veer onto the new, more up-hill, path?

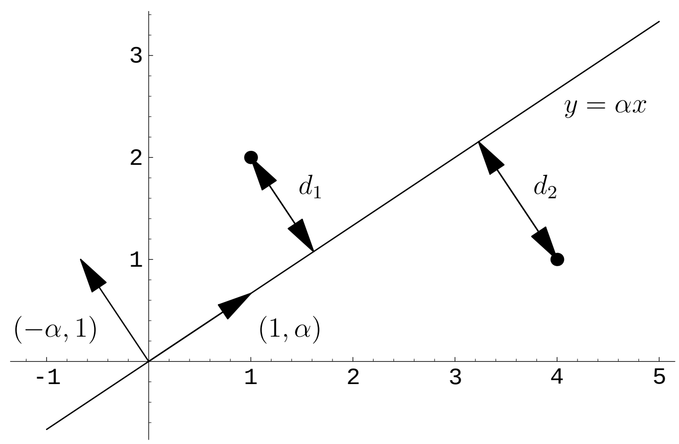
   (b) What angle does the new path make with the horizontal (i.e. with $z =$ constant in 3-space)?

   (c) If the steepest direction is to the North-West (not as shown in the picture), find a vector which is perpendicular to the surface at the point P (you should take the positive x-axist to be due East and the positive y-axis to be due North).

6. Refering to the diagram to the right, you must find α which minimizes the sum of the *perpendicular* squared distances, $f(\alpha) = d_1^2 + d_2^2$. The points are at $(1, 2)$ and $(4, 1)$.

   (a) Your first task is to find an expression for $f(\alpha)$. Hint: think about vector projection; the vectors $(1, \alpha)$ and/or $(-\alpha, 1)$ might be useful.

   (b) Find the value of α which minimizes f.



7. Suppose the $n \times n$ tridiagonal matrix

$$T = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & -1 & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}$$

has n eigenvalues λ_j with corresponding eigenvectors $\vec{v}_j$, for $j = 1, 2, \cdots, n$. Find all the eigenvalues and corresponding eigenvectors of the $n \times n$ tridiagonal matrix

$$A = \begin{bmatrix} 1+2\sigma & -\sigma & 0 & \cdots & 0 \\ -\sigma & 1+2\sigma & -\sigma & \ddots & \vdots \\ 0 & -\sigma & 1+2\sigma & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\sigma \\ 0 & \dots & 0 & -\sigma & 1+2\sigma \end{bmatrix}$$

in terms of λ_j, $\vec{v}_j$ and σ (assume $\sigma \neq 0$).

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation that satisfies

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Find a vector $\vec{x}$, such that $T(\vec{x}) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

9. Using cylindrical coordinates, find the volume of the region E in space specified by the inequalities $x^2 + y^2 \leq 2y$ and $0 \leq z \leq \sqrt{x^2 + y^2}$.

   Hint: You might want to use the integral formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

10. Find the maximum and minimum values of $f(x, y) = x^2 - 2x - y$ subject to the constraint $(x-1)^2 + y^2 = 1$.