Qualifying Exam
September 13, 2010

Directions: In all problems you must show your work in order to receive credit. You may use a calculator BUT you must explain thoroughly how you obtained your answers!

1. A spotlight on the ground shines on a wall 12 meters away. If a man 2 meters tall walks from the spotlight toward the wall at a speed of 1.6 meters per second, how fast is the length of his shadow on the building decreasing when he is 4 meters from the wall?

2. Let $\alpha \in \mathbb{R}$ and
$$f(x) = \begin{cases} \frac{1}{n^\alpha} & \text{if } x = \frac{1}{n},\ n \text{ a positive integer,} \\ 0 & \text{for all other real numbers.} \end{cases}$$
Determine, using just the definitions involved, the values of α such that

   (a) f is continuous at $x = 0$.
   (b) f is differentiable at $x = 0$.

3. A bowl is shaped like a hemisphere with diameter 30 cm. A ball with diameter 10 cm is placed in the bowl and water is poured into the bowl to a depth of h cm, so that the top of the ball is above the water level. Assuming the ball does not float, find the volume of water in the bowl.

4. Find the minimum and maximum values of $f(x, y) = 6xy - 4x^3 - 3y^2$ on the closed triangular region bounded by the lines $y = 0$, $x = 1$ and $y = 3x$.

5. Compute $\displaystyle\int_0^2 \int_{y^2}^4 y\cos(x^2)\, dxdy$.

6. (a) Find the Maclaurin series (Taylor series at 0) for $\sin x$ and $\cos x$.

   (b) Use part (a) to find real numbers A and B such that
$$\lim_{x\to 0} \frac{A\sin x - x(1 + B\cos x)}{x^3} = 1.$$

7. Assume as known that $1+x \le e^x$ for $x \ge 0$. Define the sequence $(a_n)_{n\ge1}$ recursively by
$$a_0 = 1/2, \quad a_n = \ln(1+a_{n-1}) \text{ for } n \ge 1.$$

   (a) Prove that $\lim\limits_{n\to\infty} a_n = 0$.

   (b) Find the radius of convergence of the power series $\sum\limits_{n\ge0} a_n x^n$, where $(a_n)_{n\ge1}$ is the sequence defined above.

8. Let $a, b \in \mathbb{R}\backslash\{0\}$. Consider the symmetric matrix
$$A = \begin{bmatrix} 1 & a & b \\ a & a^2 & ab \\ b & ab & b^2 \end{bmatrix}.$$

   (a) Find the kernel and the rank of A.

   (b) Find two orthogonal eigenvectors for A.

9. Let V be a 3-dimensional vector space, and $T : V \to V$ a linear transformation with $T^2 \ne 0$ but $T^3 = 0$, where $0 : V \to V$ is the zero map.

   (a) Show that there is an element $v \in V$ such that $\{v, T(v), T^2(v)\}$ is a basis for V.

   (b) Is T diagonalizable? (Think about possible eigenvalues of T.)

10. Is there a differentiable function on $[0, \infty)$ whose derivative equals its 2010th power and whose value at the origin is strictly positive?