Number: ____________________

Graduate Qualifying Exam – Spring 2014

Note: In all problems you must show your work in order to receive credit. You may use a calculator BUT you must explain thoroughly how you obtained your answers. You have **3 hours** to complete the exam.

Problem 1. Let $n \geq 1$ be a fixed integer. Define $f : (0, \infty) \to (0, \infty)$ by $f(x) = x^n e^{-x}$.
(a) Does f have any horizontal asymptotes? If yes, find them.
(b) Does f have a global maximum? If yes, find it.
(c) Find the inflection points of f.
(d) Use (b) to prove that $ex \leq ne^{x/n}$ for all $x > 0$.

Problem 2. A person 6ft tall walks 5ft/sec along one edge of a straight road 30ft wide. On the other edge of the road, ahead of the person, there is a light atop a pole 18ft high. How fast is the length of the person's shadow increasing when the person is 40ft from the point directly across the road from the pole?

Problem 3.
(a) Evaluate $\int_0^1 e^{\sqrt{x}} dx$.
(b) Find a function f such that, for all $x > 2$, we have

$$x^2 = 1 + \int_1^{2x} \sqrt{1 + [f(t)]^2} dt.$$

Problem 4. Consider the region R in the xy-plane bounded by $y^2 = 2(x - 3)$ and $y^2 = x$. Find the volume of the solid generated by rotating R around the x-axis.

Problem 5. Find the maximal area of a rectangle that is inscribed in the ellipse $(x/2)^2 + y^2 = 1$ and whose sides are parallel to the coordinate axes in the xy-plane.

Problem 6. Let $\vec{u} \in \mathbb{R}^n$ be such that $\vec{u}^T \vec{u} = 1$. Define the matrix $M := I - 2\vec{u}\vec{u}^T$; here, I denotes the $n \times n$ identity matrix and $\vec{u}^T$ denotes the transpose of the vector $\vec{u}$.
(a) Compute $M^T - M$ and M^2.
(b) Use part (a) to show that if λ is an eigenvalue of M, then $\lambda \in \{-1, 1\}$.
(c) Find an eigenvector of M corresponding to the eigenvalue -1 and an eigenvector of M corresponding to the eigenvalue 1.

Problem 7. Let $n \in \mathbb{N}$ and let V be an n-dimensional vector space over $\mathbb{C}$. Suppose $T : V \to V$ is a linear transformation which has the property that there exists an integer $m \geq 1$ such that $T^m = 0$.
(a) Show that T has an eigenvector corresponding to the eigenvalue 0.
(b) Prove, using induction, that there exists a basis for V such that the matrix of T with respect to this basis is strictly upper triangular.

Problem 8. Find the general solution of the system of differential equations

$$\begin{cases} \frac{dx}{dt} &= x + 1 \\ \frac{dy}{dt} &= x^2 - y. \end{cases}$$

Problem 9. The sequence $(a_n)_{n=1}^{\infty}$ is defined recursively by

$$a_1 = 1, \;\; a_{n+1} = \left(1 + \frac{1}{n}\right)^{-n} a_n.$$

(a) Show that (a_n) is bounded and monotonic, and compute $\lim_{n\to\infty} a_n$.

(b) Find the radius of convergence of the series $\sum_{n=1}^{\infty} a_n x^{2n}$.

Problem 10. Let $\sum_{n=1}^{\infty} x_n$ be a convergent series of positive numbers. Show that the series $\sum_{n=1}^{\infty} \cos(x_n)$ is divergent and that the series $\sum_{n=1}^{\infty} \sin(x_n)$ is convergent.