GRADUATE QUALIFYING EXAM — SPRING 2016
MONDAY, MARCH 28, 2016

Instructions. No books, notes, calculators, cellphones, etc. allowed. You may use nothing except a writing utensil, paper, and your mind. Rigorously justify all of your solutions, quoting well-known mathematical results whenever possible. Do not trivialize any problem. If you have a question, raise your hand.

Write on one side of each sheet of paper, and clearly label your solutions. Write your code name on each sheet of paper you hand in. Do not write your real name on anything you hand in.

Each problem carries equal weight.

Problem 1. Given a set of data points $\{(x_1, y_1), \ldots, (x_n, y_n)\}$, to find the least squares line $y = mx+b$ of best fit to the data, one minimizes $Q = \sum_{i=1}^{n}(\hat{y}_i - y_i)^2$ — the sum of the squares of the difference between y_i and the predicted data $\hat{y}_i = mx_i + b$. While doing the minimization to find m and b, show the following.

(a) Show that $\bar{y} = m\bar{x} + b$ where $\bar{x} = (x_1 + \cdots + x_n)/n$ and $\bar{y} = (y_1 + \cdots + y_n)/n$.
(b) Show that $\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = (\sum_{i=1}^{n} x_i y_i) - n\bar{x}\bar{y}$.
(c) Show that $m = (\Delta\vec{x}\cdot\Delta\vec{y})/(\Delta\vec{x}\cdot\Delta\vec{x})$ where $\vec{x} = (x_1 - \bar{x}, \ldots x_n - \bar{x})$ and $\vec{y} = (y_1 - \bar{y}, \ldots, y_n - \bar{y})$.

Problem 2. (Gabriel's Horn) By rotating about the x-axis, the region bounded by $y = 1/x$, the x-axis and $x \geq 1$, show that one obtains a solid of revolution that has finite volume but infinite surface area. Thus, paradoxically, we could fill the volume with a finite amount of paint but not be able to paint its surface! Recall the formula for arc length of the graph $y = f(x)$ is $\int_a^b \sqrt{1 + f'(t)^2}dt$.

Problem 3. For $x > 0$, give a rigorous proof that

$$\ln(1 + x) < \frac{x}{\sqrt{1 + x}}.$$

Problem 4. Determine whether each of the three series

$$\sum_{n=1}^{\infty}(1/2 - 1/n)^n, \quad \sum_{n=0}^{\infty}\frac{n^n}{n!}, \quad \sum_{n=2}^{\infty}\frac{1}{n\log(n)},$$

converges or diverges.

Problem 5. Let s_n be the sequence of numbers defined recursively as

$$s_1 = 1, \quad s_{n+1} = \sqrt{6 + s_n}.$$

For example, $s_2 = \sqrt{7}$. Show that for all integers $n \geq 1$, $s_n \leq s_{n+1}$ and $s_n \leq 10$. Explain why $L = \lim_{n\to\infty} s_n$ exists and compute the exact value of L.

Problem 6. Suppose the temperature at each point of a plate is given by $T(x,y) = -x^2 + y^2 + 2xy$ and that a bug's position at time t is given by $\gamma(t) = (t-1, t^2-2)$.

(a) Sketch the path of the bug for $t \geq 0$.
(b) What is the velocity of the bug as it goes through the point $(1,2)$?
(c) Find the rate of change with respect to time of the temperature of the bug as it goes through the point $(1,2)$.
(d) In what direction should the bug move from $(1,2)$ so as to best avoid the heat?
(e) If the bug likes the temperature at the point $(1,2)$, in what direction should it go to maintain this temperature?

Problem 7. Let $f : (-2,2) \to \mathbf{R}$ be defined by the rule

$$\begin{cases} x^2 & x \in \mathbf{Q}, \\ 2x-1 & x \notin \mathbf{Q}. \end{cases}$$

Show that f is differentiable only at $x = 1$ and compute $f'(1)$.

Problem 8. Suppose that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent in a vector space V. Working directly from the definition of linear independence, determine if

$$\{\mathbf{u}-\mathbf{v}, \mathbf{v}-\mathbf{w}, \mathbf{u}-\mathbf{w}\}$$

is linearly independent.

Problem 9. Let $g : \mathbf{R} \to \mathbf{R}$ be defined as $g(0) = 1$ and $g(t) = \sin(t)/t$ $(t \neq 0)$. Clearly explain why the function $F : (0,\infty) \to \mathbf{R}$ defined by

$$F(x) = \int_0^{\sin(x)} x g(t) dt$$

is differentiable. Compute the derivative $F'(\pi)$.

Problem 10. Determine the most general function $y : \mathbf{R} \to \mathbf{R}$ that satisfies the differential equation

$$y'' - 4y' + 5y = 0$$

Express your answer as a real valued function of a real variable.