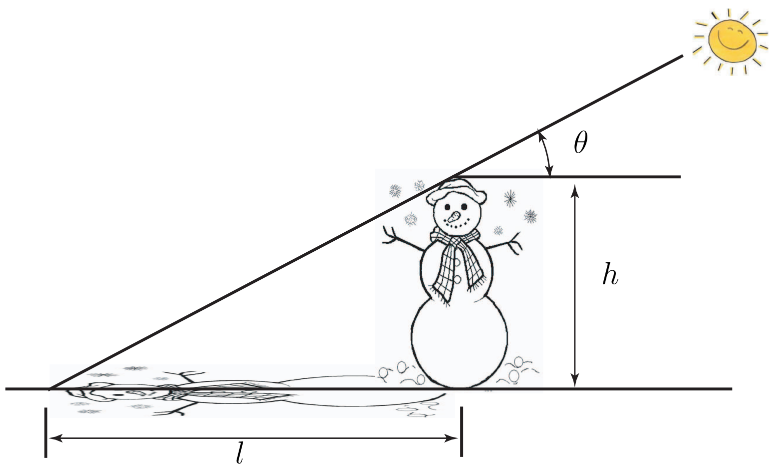
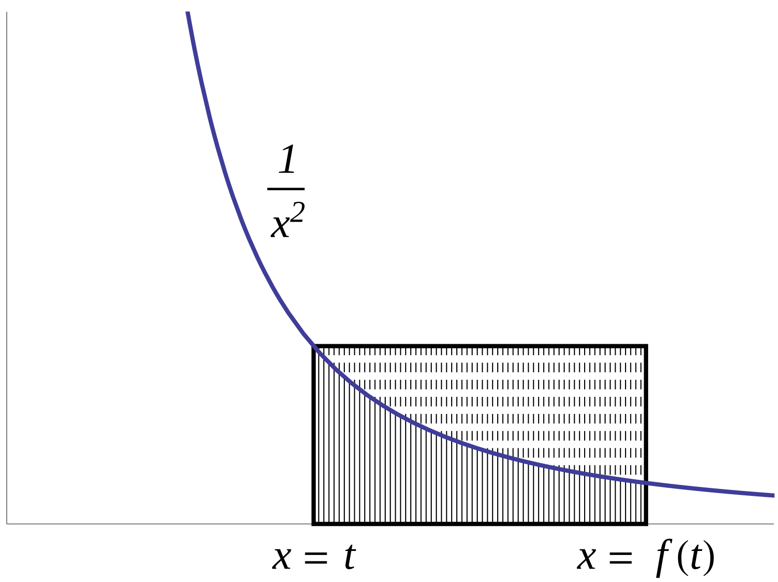
Graduate Qualifying Exam – Fall 2011 – Monday September 12

There are **10 questions**, each worth 10 points. You may use calculators. Be sure to show your working.

1. (10 pts) A snowman is melting as the afternoon sun sets. The sun sets at a constant rate so that θ decreases by $\pi/2$ radians in 6 hours; the snowman melts at a rate so that its height h decreases at a rate of $1/2$ cm per minute. Find the rate at which l, the length of the snowman's shadow, is changing at the instant when $h = 1.5$ meters and $\theta = \pi/6$ radians.



2. (10 pts) Find the function $f(t)$ so that, for all $t > 0$, the area trapped between $x = t$, $x = f(t)$, (above) $y = 0$ and (below) $y = \dfrac{1}{x^2}$ is *equal* to the area trapped between $x = t$, $x = f(t)$, (below) $y = \dfrac{1}{t^2}$ and (above) $y = \dfrac{1}{x^2}$. With respect to the picture shown, find $f(t)$ so that the two areas depicted are equal, for all $t > 0$. Note: a trivial solution is $f(t) = t$; find the non-trivial solution.



3. For 12 hours each day, a factory pumps a chemical solution into a 12,000 liter reservoir. The solution enters the reservoir at a rate of 1000 liters per hour, and the amount of chemical in the solution at time t hours from the beginning is αt kilograms, where α is a constant. Let the amount of chemical in the reservoir at the beginning of the 12 hours be A_0 kilograms.

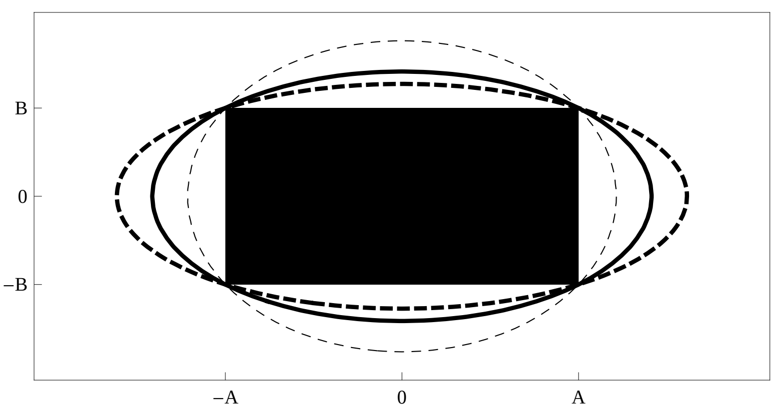
   The contaminated solution mixes well (instantly) in the reservoir and the well-mixed solution flows out of the reservoir at a rate of 1000 liters per hour.

   (a) (7 pts) Find $A(t)$, the amount of chemical in the reservoir at time t for $0 \le t \le 12$. (Your answer will be in terms of α and A_0.)

   Suppose for the remaining 12 hours of the day the factory pumps 1000 liters per hour of pure clean water into the reservoir, which mixes well and flows out again at 1000 liters per hour. The amount of chemical in the reservoir t hours into this cleansing cycle is $B(t) = A(12)e^{-t/12}$ (where $A(12)$ comes from (a)).

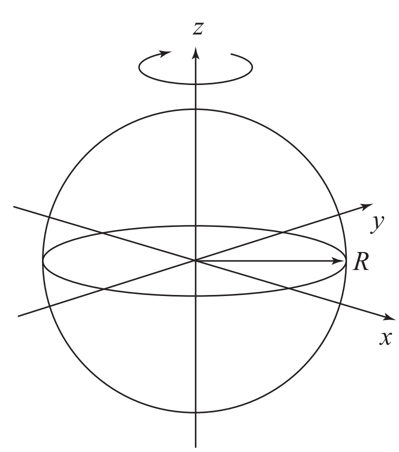
   (b) (3 pts) Determine the constant α which results in the amount of chemical in the reservoir at the end of the 12 hour cleansing cycle being equal to the amount at the beginning of the day, A_0. (Your answer will be in terms of A_0.)

4. (10 pts) Consider the family of all ellipses $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; their areas are πab. Find the pair (a, b) ($a > 0$, $b > 0$) which minimizes the area of the ellipse which passes through the corners of the rectangle of side-lengths $2A$ by $2B$ centered at the origin. A and B are fixed constants. Three such ellipses are shown in the figure.



5. (10 pts) Find the mass of fluid contained in a spherical centrifuge (a sphere of radius R spinning about its axis; see figure) if the fluid density is given by

$$\rho(x, y, z) = |z| \exp\left[-\left(1 - \frac{x^2 + y^2}{R^2}\right)^2\right].$$



6. Suppose the concentration of blood in parts per million of water can be approximated by

$$C(x, y) = \exp\left[-\left(2x^2 + 3y^2\right)/10^4\right],$$

where x and y are the coordinates in meters from the blood source. Sharks that detect the presence of blood will respond by moving continually in the direction of the strongest scent. Suppose a shark detects a blood scent at $x = x_0 \neq 0$, $y = y_0$, and let $y(x)$ describe the path that the shark follows as it moves toward the blood source (which is at $(0, 0)$).

(a) (5 pts) Find an expression for $\frac{dy}{dx}$, and then the curve $y(x)$.

(b) (5 pts) Use your answer in part (a) to find the distance the shark will travel to the source.

7. (a) (4 pts) Determine the limit of the series $\sum_{n=2}^{\infty} \log\left(1 - \frac{1}{n^2}\right)$, or else show that it diverges. (Hint: Expand the logarithm and consider a series of differences.)

(b) (6 pts) Find the interval of convergence of

$$\sum_{n=2}^{\infty} \frac{(x+1)^n}{(n \log n) 3^n},$$

and determine at each endpoint of the interval whether the series converges or diverges.

8. (a) (5 pts) Fact: If x is not an odd multiple of of $\pi/2$ then $f(x) = \sum_{n=0}^{\infty} \sin^n x = \frac{1}{1 - \sin x}$. Use this fact to determine the limit of the infinite series $\sum_{n=1}^{\infty} n \sin^n x$, and state where the series does/does not converge.

(b) (5 pts) Find the MacLaurin series (the Taylor series based at $x = 0$) for $g(x) = \ln\sqrt{\frac{1-x}{1+x}}$, $|x| < 1$. (Hint: Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.)

9. Let $B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

(a) (3 pts) Diagonalize B.

(b) (4 pts) A real symmetric matrix A is said to be *positive definite* if $x^T A x > 0$ for every vector $x \neq 0$. Is matrix B positive definite? Justify your answer. (Note: this can be done without using the diagonalization of part (a).)

(c) (3 pts) Let λ be an eigenvalue of an $n \times n$ symmetric positive definite matrix. Prove that $\lambda > 0$.

10. (10 pts) Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V and $n \geq 2$. For what values of n is the set

$$\{v_1 + v_2, v_2 + v_3, \ldots, v_{n-1} + v_n, v_n + v_1\}$$

also a basis for V? Justify your answer.