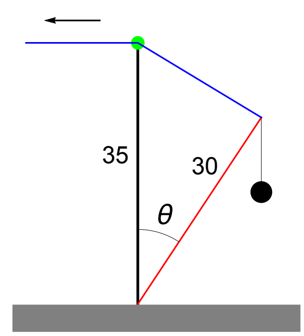
Qualifying Exam: Fall 2015

1. The length of the curve which is the graph of a function $y = f(x)$ between $x = a$ and $x = b$ is equal to $\int_a^b \sqrt{1 + f'(x)^2}\, dx$. Consider the curve given by the graph of $y = f(x) = 2\sqrt{x}$. Let $a(t) = t$ and $b(t) = 2t + 1$. Let $L(t)$ be the length of the curve given by the graph of $f(x)$ between $x = a(t)$ and $x = b(t)$.

   Find t^*, the value of $t > 0$ where the length $L(t)$ is **minimal**. Prove that it is indeed a minimum.

2. The device shown in the picture is called a "derrick" – the vertical $35m$ pole (black) is fixed and has a pulley at the top (green). A cable (blue) runs over the pulley and is attached to the end of the $30m$ boom (red) which is free to rotate about its lower end. There is a weight hanging from the end of the boom on a cable of fixed length.

   If the (blue) cable is drawn at the rate of $2m/s$ to lift the boom, how quickly is the weight being raised *vertically* at the instant that θ, the angle between the pole and the boom, is $12°$?
   In case you've forgotten,



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$
$$a^2 = b^2 + c^2 - 2BC\cos A,$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B,$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

3. (a) Suppose that the MacLaurin series (Taylor series based at 0) for the function

$$f(x) = \frac{1+x}{1+ax} - \ln(1+bx)$$

   is such that the coefficients of x and x^2 are both zero. Find two possible pairs of the values of the constants a and b. Verify that for one of these pairs of values, in fact, all coefficients of powers of x in the series expansion of $f(x)$ are zero.

   (b) Prove or disprove: $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = 0.$

4. (a) Prove that $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} na_n = A > 0$. Hint: Think harmonic series.

   (b) Using part (a), or some other method, show that $\sum_{n=1}^{\infty} \ln\left(1 + \frac{2}{n}\right)$ diverges.

   (c) Determine whether or not $\sum_{n=1}^{\infty} \frac{n^{2n}}{7^n (n!)^2}$ converges.

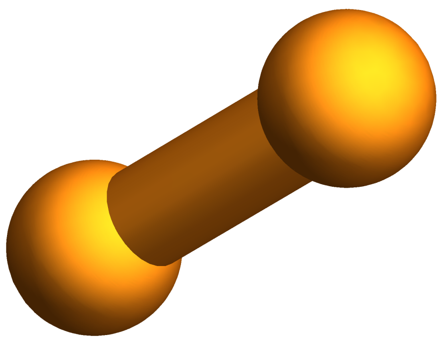
5. Let A be an $n \times n$ real matrix.

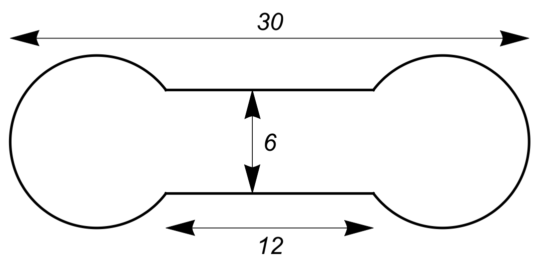
   (a) Show that if $A^T = -A$ and n is odd, then $\det A = 0$.

   (b) Let $C = BA$ for some nonsingular matrix B. Show that $\text{rank}(C) = \text{rank}(A)$.

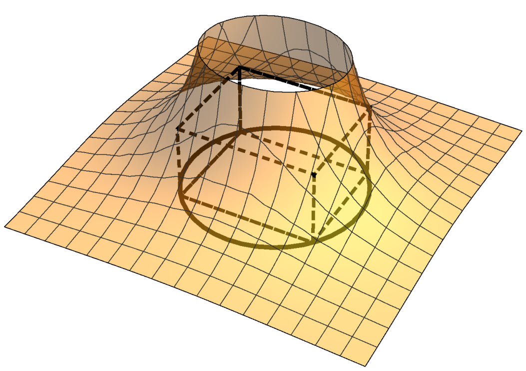
   (c) Let A be an **orthogonal** matrix. Show that $\text{Range}(I - A) \subseteq \big(\text{Null}(I - A)\big)^{\perp}$, that is, every element in the range (image) of $I - A$ is perpendicular to every element in the null space (kernel) of $I - A$.

6. Shown is a dog's toy made from two (partial) balls attached onto a (solid) cylindrical rod; the dimensions (in cm) are given in the cross-section. Compute the volume of the toy.





7. Consider a rectangular box whose bottom corners lie on the unit circle centered at the origin in the xy-plane, and whose top corners meet the surface $f(x, y) = \dfrac{1}{2x^2 + 3y^2}$, and whose sides are parallel to the coordinate axes. An example of such a rectangular box is shown in the figure. Find the **volume** of the box which is **maximal** among all such boxes.



8. Let $\mathcal{A}$ be a linear transformation, $\mathcal{A} : V \to V$, where V is an n-dimensional vector space. Suppose that $\vec{x} \in V$ is a non-zero vector such that $\mathcal{A}^{n-1}\vec{x} \neq \vec{0}$, but $\mathcal{A}^{n}\vec{x} = \vec{0}$.

   (a) Show that $\{\vec{x}, \mathcal{A}\vec{x}, \mathcal{A}^2\vec{x}, \ldots, \mathcal{A}^{n-1}\vec{x}\}$ is a linearly independent set.

   (b) Recall that an eigenvalue of a linear transformation T is a scalar λ for which the equation $T\vec{v} = \lambda\vec{v}$ has a non-zero solution $\vec{v}$. What are the eigenvalues of $\mathcal{A}$? Hint: Note that the set in part (a) forms a basis for V.

9. Consider the first order ordinary differential equation (ODE)

$$\frac{dy}{dx} + x|x|y = x|x|, \quad y(0) = y_0.$$

   The theory of ODEs guarantees the existence and uniqueness of the solution for any initial condition y_0. You may take it as a fact that, for all integers $n > 1$, $x^n|x|$ is differentiable and $\dfrac{d}{dx}x^n|x| = (n+1)x^{n-1}|x|$.

   (a) Without solving the ODE, show that if $y(x)$ is a solution, then $x = 0$ is a local maximum for $y_0 > 1$ and a local minimum for $y_0 < 1$. (Hint: use the ODE to compute $y'(0)$, and to determine the sign of $y'(x)$ for x near $x = 0$.)

   (b) Solve the initial value problem. Note: it does not suffice to use your solution to (b) to answer (a) directly.

10. Every parabola has a *focal point*: consider a ray parallel to the axis of the parabola meeting the parabola (red ray), then reflecting (blue ray) so that the angle between the reflected ray and the tangent line at the point of intersection (blue angle) is equal to the angle between the incoming ray and the same tangent line (red angle). See the picture. The focal point is on the axis of the parabola, and all such reflected rays meet at this point.

    Let the parabola be $y = x^2$; let the focal point be $(0, \varphi)$. Find φ. Your derivation must also prove that φ is correct. (Hint: use vector calculus and the dot product.)