Qualifying Examination Solutions 12 September 2016

1. A straight road follows a diagonal of a square city of side 10 miles. The density of the population at distance x miles from the road is well-approximated by $\rho(x) = 15 - 3x/\sqrt{2}$ thousand people per square mile. Find the total population of the city.

Solution. The total population T of the city is given by

$$T = 1000 \int_0^{5\sqrt{2}} 2\rho(x)(10\sqrt{2} - 2x)\, dx = 6000\sqrt{2} \int_0^{5\sqrt{2}} (5\sqrt{2} - x)^2\, dx = 1000000.$$

2. Consider the function

$$f(x) = \arcsin\sqrt{x} - \arctan\left(\sqrt{\frac{x}{1-x}}\right).$$

(a) What is the domain of f?

(b) Calculate the derivative of f.

(c) Sketch the graph of f.

Solution.

(a) We need $x \geq 0$ and $\frac{x}{1-x} \geq 0$ so that $x < 1$. Consequently, the domain of x is $[0, 1)$.

(b) Several applications of the chain rule yield $f'(x) = 0$ for $x \in (0, 1)$.

(c) By inspection, $f(0) = f(1/2) = 0$, so that f is identically zero on its domain.

3. A small triangular island occupies the region bounded by the y-axis, the line $y = -2$, and the line $y = 2 - x$. The temperature at each point (x, y) on the island is given by $T(x, y) = 60 + x^2 + xy + 2y^2 - 3x + 2y$. Find the distance between the hottest and coldest points on the island.

Solution. We have

$$\operatorname{grad} T(x, y) = (2x + y - 3, x + 4y + 2),$$

so that $\operatorname{grad} T = 0$ when $2x + y = 3$ and $x + 4y = -2$, i.e., when $x = 2$ and $y = -1$. Since $T(2, -1) = 56, T(0, -2) = 64, T(4, -2) = 60, T(0, 2) = 72$, and also $T(0, y) = 60+2y(y+1) \geq 59.5, T(x, -2) = 64+x^2-5x \geq 57.75, T(x, 2-x) = 72+2x^2-11x \geq 56.875$, we see that the hottest point of the island is at $(0, 2)$ and the coldest at $(2, -1)$. These points lie at distance $\sqrt{13}$ from each other.

4. A wooden object is a ball in $\mathbb{R}^3$ of radius R, centered at $(0,0,0)$, with the part lying in the infinite cylinder $\{(x,y,z) : x^2+y^2 < r^2\}$ removed, where $r < R$. (In other words, a hole of radius r has been drilled through the center of the ball to make the object). Find the volume $V(R,r)$ of the object in terms of R and r.

Solution. We have

$$V(R,r) = \pi \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left(\left(\sqrt{R^2-h^2} \right)^2 - r^2 \right) dh = \frac{4\pi}{3} \left(R^2-r^2\right)^{3/2}.$$

5. A solid tetrahedron T has vertices $(3,4,5),(3,4,6),(4,5,5)$ and $(4,6,5)$.

(a) Find the volume of T.

(b) Find the surface area of T.

Solution.

(a) Write $A=(3,4,5), B=(3,4,6), C=(4,5,5)$ and $D=(4,6,5)$. The base ACD of T in the plane $z=5$ has area $1/2$, and the distance from B to the plane $z=5$ is 1. Therefore, the volume of T is $1/6$.

(b) With the above notation, the areas of triangles ACD, ABD and ABC are easily seen to be $1/2, \sqrt{5}/2$ and $1/\sqrt{2}$ respectively. Triangle BCD is a little trickier. Its base CD has length 1, and point B lies at a perpendicular distance $\sqrt{2}$ from the line containing CD, so that triangle BCD has area $1/\sqrt{2}$. Consequently, the full surface area of T is $(1+\sqrt{5}+\sqrt{8})/2$.

Alternatively, one can make use of Heron's formula for the area of a triangle in terms of its side lengths.

6. Let $x_0 \in \left(0, \frac{\pi}{2}\right)$. Define the sequence $\{x_n\}$ by $x_n = \sin x_{n-1}$ for $n \geq 1$.

(a) Prove rigorously that $\lim_{n\to\infty} x_n = 0$.

(b) Find the limit $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$. Justify your answer.

Solution.

(a) Since $\sin x \leq x$ for $x \geq 0$, $\{x_n\}$ is a decreasing sequence bounded below by 0, so that $x_n \to \ell$ for some $\ell \geq 0$. Suppose that $\ell > 0$. Write $\varepsilon = \ell - \sin \ell$. Note that $f(x) = x - \sin x$ is an increasing function of x, so that $\varepsilon < (\ell+\varepsilon) - \sin(\ell+\varepsilon)$, i.e., $\sin(\ell+\varepsilon) < \ell$. Now choose $\delta > 0$ so that $\sin x \in (\sin\ell - \varepsilon, \sin\ell+\varepsilon)$ whenever $x \in (\ell-\delta, \ell+\delta)$. Since $x_n \to \ell$, there exists m such that $x_m < \ell+\delta$, so that

$$x_{m+1} = \sin x_m < \sin(\ell+\delta) < \sin\ell + \varepsilon < \ell,$$

which is a contradiction, since $\{x_n\}$ is decreasing. Therefore $\ell = 0$.

(b) We have

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\sin x_n}{x_n} = 1,$$

since $\frac{\sin x}{x} \to 1$ as $x \to 0$ and $x_n \to 0$ as $n \to \infty$. In detail, given $\varepsilon > 0$, we choose $\delta > 0$ so that $\frac{\sin x}{x} \in (1-\varepsilon, 1+\varepsilon)$ whenever $0 < x < \delta$, and then choose N_δ so that $0 < x_n < \delta$ whenever $n \geq N_\delta$. Then, if $n \geq N_\delta$, we will have

$$\left| \frac{x_{n+1}}{x_n} - 1 \right| = \left| \frac{\sin x_n}{x_n} - 1 \right| < \varepsilon.$$

7. Determine whether the integral

$$\int_1^\infty \frac{9x^2 + 11}{\sqrt{x^5 + 14x^4 - 2\cos(x^3)}}\, dx$$

converges or diverges. Give a careful and complete justification for your answer, stating any theorems you use, and verifying that their hypotheses are satisfied when you apply them.

Solution. The integral diverges by comparison with $\int_1^\infty x^{-1/2}\, dx$. Specifically, for $x \geq 15$, we have $x^5 + 14x^4 - 2\cos(x^3) \leq x^5 + 14x^4 + 2 < x^5 + 15x^4 \leq 2x^5$, so that

$$\frac{9x^2 + 11}{\sqrt{x^5 + 14x^4 - 2\cos(x^3)}} \geq \frac{9x^2}{\sqrt{2x^5}} > 6x^{-1/2}.$$

(We are using the theorem that if f and g are two positive Riemann-integrable functions such that $f(x) \geq g(x)$ for all $x \geq a$, and $\int_a^\infty g(x)\, dx$ diverges, then $\int_a^\infty f(x)\, dx$ diverges.)

8. Consider the family of matrices

$$A_x = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

For which values of x can A_x be diagonalized? Justify your answer.

Solution. The eigenvalues of an upper-triangular matrix lie on the main diagonal, so A_x always has eigenvalues 1 and 2, each with multiplicity 2. The eigenspace corresponding to the eigenvalue 2 is

$$\text{span}\left\{ \begin{pmatrix} x \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

while the eigenspace corresponding to the eigenvalue 1 is

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

when $x \neq 0$, and

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\},$$

when $x = 0$. Therefore, A_x is only diagonalizable when $x = 0$.

9. Find all real 2×2 matrices A satisfying $A^2 = I$. Give your answer as an explicit formula.

Solution. The determinant $\det A$ of A is either 1 or -1. If $\det A = 1$, then, since $A = A^{-1}$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

so that $b = c = 0$ and $a = d$, which, together with $\det A = 1$, yield $A = \pm I$. If, however, $\det A = -1$, then we instead have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix},$$

yielding $a = -d$ and, since $\det A = -1$, $a^2 + bc = 1$. Consequently, we obtain

$$A = \begin{pmatrix} \pm\sqrt{1-bc} & b \\ c & \mp\sqrt{1-bc} \end{pmatrix},$$

for all b, c such that $bc \leq 1$, together with the two solutions $A = \pm I$ described above.

10. A large and prosperous country is surrounded by a giant wall. Its majestic demagogue has decreed that its growing population $P(t)$ must never exceed $400,000,000$. Accordingly, the rate of growth of the population at any time is proportional to the product of the population at that time and the additional population permitted by the demagogue at that time.

(a) Write down a differential equation for $P(t)$.

(b) One day, the demagogue's demographer reports that $P(t)$ is passing through a point of inflection, for which he is immediately fired. Calculate the population of the country when this happens.

Solution.

(a) Writing L for 400,000,000, the differential equation is

$$\frac{dP}{dt} = kP(L - P),$$

where k is a positive constant.

(b) There are two ways to answer this question. One is to solve the differential equation by separation of variables. Alternatively, one can directly differentiate the original differential equation to get

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt}(L - P) - kP\frac{dP}{dt} = k\frac{dP}{dt}(L - 2P),$$

which is zero when either $dP/dt = 0$ or $P = L/2$. Since $dP/dt = kP(L-P) > 0$, we must have $P = L/2$; furthermore, d^2P/dt^2 is positive when $P < L/2$, and negative when $P > L/2$. Therefore, the population at the point of inflection is 200,000,000.

(Note: the general solution of the differential equation is

$$P(t) = \frac{LAe^{Lkt}}{1 + Ae^{Lkt}},$$

for a suitable constant A, to be determined by the initial conditions.)