Qualifying Exam March 30, 2009 Number ________________

EXPLAIN ALL YOUR ANSWERS AND SHOW ALL YOUR WORK. NO CALCULATORS.

1. The matrix

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

has characteristic polynomial $-\lambda(\lambda - 1)^2$.
(i) Diagonalize A.
(ii) Find A^{17}, justifying your answer.

2. Let R be the two-dimensional region in the first quadrant of the xy-plane above the x-axis, below the line $y = x - 1$, and inside the ellipse $x^2 + 5y^2 = 9$. Find the volume of the three-dimensional region obtained by rotating R around the y-axis.

3. Find the equation, in the form $z = ax + by + c$, of the plane tangent to the surface

$$z^3 + z^2 = x^2 + y^2 - 3$$

at the point $(2, 1, 1)$.

4. (i) Find the Maclaurin series (Taylor series at 0) for $\cos x$.
(ii) Consider the integral

$$\int_0^1 \frac{1 - \cos x}{x^2}\, dx.$$

(a) Use your answer from part (i) to find the Maclaurin series for the integrand.
(b) Find an approximation to the integral, with an error of less than 1/200. (Give your answer as a rational number. You may assume that the Maclaurin series you obtained in part (ii)(a) is equal to the integrand for all x.)

5. Prove that

$$\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+4} + \frac{n}{n^2+9} + \cdots + \frac{n}{n^2+n^2} \right) = \frac{\pi}{4}.$$

(Hint: What is $f(i/n)$ when $f(x) = 1/(1 + x^2)$?)

6. In what follows all quantities are hourly averages (and thus may have non-integer values). The number of patients seen at a clinic, P, depends on the quantity of equipment, Q, the number of doctors, M, and the number of nurses, N, according to

$$P(Q, M, N) = Q^{0.1} M^{0.6} N^{0.3}.$$

In dollars per hour, the cost of equipment is 20, the cost of a doctor is 40, and the cost of a nurse is 20. Use the method of Lagrange multipliers to find the combination of equipment, doctors, and nurses that maximizes the number of patients seen at the clinic assuming that the total amount spent is 200 dollars per hour.

7. A function f is defined by

$$f(x) = \int_0^x \sin(xt)\,dt.$$

(i) Calculate $f(1)$ and $f(2\pi)$.
(ii) Find $f'(x)$.
(iii) Calculate $f'(0)$ and $f'(1)$.

8. For each of the following, state whether it is true or false. (You can write "T" or "F.") Justify your decision with a brief proof or an example, as appropriate.
(i) The kernel of a linear transformation is always a subspace.
(ii) The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.
(iii) A linear transformation T is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution $x = 0$.
(iv) All orthogonal matrices are symmetric.
(v) For $n \geq 2$, multiplication of two $n \times n$ upper triangular matrices is commutative.

9. A tank with a capacity of 500 gallons originally contains 200 gallons of water with 100 lb. of salt in solution. Water containing 1 lb. of salt per gallon is entering at the rate of 3 gallons per minute, and the mixture is allowed to flow out of the tank at a rate of 2 gallons per minute. Find the concentration $C(t)$ of salt in the tank at any time t until it overflows.

10. A ball rolls down from the highest point on a plank whose angle with the horizontal is α, where $0 < \alpha < \pi/2$. The plank has length L. Gravity acts downward and the gravitational constant is g. Find the angle that results in the ball having the maximum *horizontal component* of velocity at the time it reaches the bottom of the plank.