Solutions for the Qualifying Exam – Fall 2012

September 17, 2012

1. Find $y'(x)$ if $y(x) = \displaystyle\int_1^{x^2} \frac{\sin(tx)}{t}\,dt$.

   Solution: Using the fundamental theorem of Calculus and the chain rule for functions of two variables:

$$\frac{d}{dx}\int_1^{x^2} \frac{\sin(tx)}{t}\,dt = \frac{\sin\left(x^3\right)}{x^2}\cdot 2x + \int_1^{x^2} \cos(tx)\,dt = \frac{3}{x}\sin\left(x^3\right) - \frac{1}{x}\sin x.$$

2. A solid ball has a radius of 0.5 meters and weighs 10 kg. It is floating in a large pool of water. How deeply is it submerged at its deepest point? (Recall the Archimedean principle: For a floating object, the weight of the displaced water equals the weight of the object. The density of water is approximately 1000 kg/m^3.)

   Solution: If the ball is submerged to the depth h, the volume of the submerged portion is given by $V(h) = \displaystyle\int_{0.5-h}^{0.5}\int_{D_{r(z)}} dA\,dz$, where $D_{r(z)}$ is a disk centered at $(0,0)$ of radius $r(z) = \sqrt{\frac{1}{4} - z^2}$. One finds $V(h) = \pi\left(\frac{h^2}{2} - \frac{h^3}{3}\right)$.

   The weight of displaced water is $1000V(h)$, so the Archimedean principle translates to the equation $1000\pi\left(\frac{h^2}{2} - \frac{h^3}{3}\right) = 10$. The solution is $\boldsymbol{h} = \mathbf{0.0820}$ meters.

3. Consider three tanks, labeled A, B and C. Initially, tank A contains 20 gal of water with a salt concentration of 0.2 lb/gal. Both tanks B and C initially contain 10 gal of pure water without any salt. Water is flowing from tank A to tank B at the rate of 0.1 gal/min, from tank A to tank C at the rate of 0.1 gal/min, and from tank B to tank C at the rate of 0.2 gal/min. In addition, water is leaking out of tank C at the rate of 0.1 gal/min. Assume perfect mixing, that is, the salt solution mixes effectively instantly in the tanks.

   What is the salt concentration in tank C when its maximum capacity of 20 gal is reached?

   Solution: Let $x(t)$ denote that the weight of the salt in tank C, and $y(t)$ the weight of the salt in tank B. The equation for $y(t)$ is $y' = -0.2\frac{y}{10-0.1t} + 0.02$. Solving this using the method of integrating factor gives

$$y(t) = 0.2(10-0.1t) - 0.02(10-0.1t)^2.$$

   The equation for $x(t)$ is $x' = 0.02 + 0.2\frac{y(t)}{10-0.1t} - 0.1\frac{x}{10+0.2t}$. Using the explicit form for $y(t)$ derived above and again the method of the integrating factor gives

$$x(t) = 0.2(10+0.2t)^{3/2} - 0.004(40-0.2t)(10+0.2t)^{3/2} - 0.04\cdot 10^{3/2}.$$

   This gives $\boldsymbol{x(t=50) = 1.317}$ lb salt in tank C after 50 minutes.

4. The temperature at any point of a flat plate is given by $T = 100 - 0.09x^2 - 0.16y^2$, where x and y are the vertical and horizontal distances from a fixed point $(0, 0)$, measured in feet, and T is measured in degrees Fahrenheit. Consider the point $(5, 2)$.

   (a) In what direction must a bug move from $(5, 2)$ in order for temperature to decrease at the fastest rate? What is this rate (in degrees per foot)?

   (b) If the bug moves at 2ft/min in the above direction, how fast is the temperature felt by the bug decreasing (in degrees per minute)?

   (c) In what direction from $(5, 2)$ must the bug move so that the temperature neither increases nor decreases?

   (d) Another bug is moving along the curve of constant temperature, starting at $(5, 2)$ in the direction found in 4c. It is moving at a constant speed of 3ft/min. Determine if the bug will return to $(5, 2)$, and if so, how long it will take.

Solution: We have $\nabla T(x, y) = (-0.18x, -0.32y)$.

(a) The direction is opposite the direction of the gradient, so $-\nabla T(5, 2) = (0.9, 0.64)$. The rate is $|\nabla T(5, 2)| \approx 1.032$ degree/ft.

(b) It's $|\nabla T(5, 2)| \cdot 2\text{ft/min} \approx 2.065$ degree/min.

(c) In the direction perpendicular to $\nabla T(x, y)$, f.ex. in the direction $(0.64, -0.9)$.

(d) It moves on the level curve $100 - 0.9x^2 - 0.16y^2 = 100 - 0.09 \cdot 5^2 - 0.016 \cdot 2^2 \stackrel{\text{def}}{=} 100 - r^2$. This is an ellipse; in particular the bug will eventually return to its initial position. It can be parametrized via

$$x = \frac{r}{0.3}\cos t, \qquad y = \frac{r}{0.4}\sin t, \qquad 0 \le t \le 2\pi.$$

The circumference is

$$\ell = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\, dt = \int_0^{2\pi} r\sqrt{\frac{\sin^2 t}{(0.3)^2} + \frac{\cos^2 t}{(0.4)^2}}\, dt \approx 31.313$$

The corresponding time is $\ell/3$ ft$\approx$ **10.44** minutes.

5. Does the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} \exp\left(\int_0^{2n\pi} |\cos(nx)|\, dx\right)$ converge or diverge? If it converges, find its limit.

Solution: Compute the integral first. Using the substitution $u = nx$ yields

$$\int_0^{2n\pi} |\cos(nx)|\, dx = \frac{1}{n}\int_0^{2n^2\pi} |\cos x|\, dx = n\int_0^{2\pi} |\cos x|\, dx = 4n.$$

Thus the series is

$$\sum_{n=0}^{\infty} \frac{1}{n!} \exp\left(\int_0^{2n\pi} |\cos(nx)|\, dx\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\exp(4n) = \sum_{n=0}^{\infty} \frac{(e^4)^n}{n!} = \boldsymbol{e^{e^4}}.$$

6. Answer the following:
   (a) Define $\sum_{n=1}^{\infty} a_n$.
   (b) Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.
   (c) Find the following limits and justify your answers rigorously. (Detailed $\varepsilon - \delta$ arguments are not necessary, but clearly explain the method you used.)
      (i) $\displaystyle\lim_{n\to\infty} \frac{1000^n}{n!}$
      (ii) $\displaystyle\lim_{n\to\infty} \left(1+\frac{k}{n}\right)^{n^2}$ (Hint: Consider the cases $k>0$, $k<0$ and $k=0$ separately.)
      (iii) $\displaystyle\lim_{n\to\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3$

Solution:

(a) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$, where $S_n = \sum_{k=1}^{n} a_k$.

(b) We have $\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$, so $0 = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} a_n$.

(c) Solutions:

(i) We know $\sum_{n=0}^{\infty} \frac{1000^n}{n!} = e^{1000}$ converges, so by (b), this implies $\displaystyle\lim_{n\to\infty} \frac{1000^n}{n!} = 0$.

(ii) We have by l'Hospital's rule

$$\lim_{n\to\infty} n^2 \ln\left(1+\frac{k}{n}\right) = \lim_{n\to\infty} \frac{\left(1+\frac{k}{n}\right)\left(-\frac{k}{n^2}\right)}{-2/n^3}$$

$$= \lim_{n\to\infty} \frac{nk}{2}\left(1+\frac{k}{n}\right) = \begin{cases} +\infty & \text{if } k>0 \\ -\infty & \text{if } k<0 \\ 0 & \text{if } k=0 \end{cases}.$$

(iii) Considering Riemann sums of width $\Delta x = 1$ (make sketch!) yields

$$\frac{1}{n^4}\int_0^n x^3\,dx \le \frac{1}{n^4}\sum_{k=1}^{n} k^3 \le \frac{1}{n^4}\int_0^{n+1} x^3\,dx,$$

so

$$\frac{1}{4} \le \frac{1}{n^4}\sum_{k=1}^{n} k^3 \le \frac{1}{4}\left(1+\frac{1}{n}\right)^4.$$

The squeeze theorem now yields $\displaystyle\lim_{n\to\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{1}{4}$.

7. Answer the following:
   (a) What is the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} nx^n$?

(b) What is the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n+1}$?

(c) By differentiation or integration or some other process find the limits of the above series.

(d) How many terms are necessary to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ within an error of 0.0001?

Solution:

(a) Use the ratio test:

$$\left|\frac{(n+1)x^{n+1}}{nx^n}\right| \to |x|.$$

So the radius of convergence is 1. At $x = 1$ and $x = -1$, the series diverges (divergence test), and so the interval of convergence is $(-1, 1)$.

(b) Use again the ratio test:

$$\left|\frac{x^{n+1}/(n+2)}{x^n/(n+1)}\right| \to |x|.$$

So the radius of convergence is 1. At $x = 1$, the series diverges (harmonic series), and at $x = -1$, it converges (alternating series). So the interval of convergence is $[-1, 1)$.

(c) $\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = x\frac{d}{dx}\sum_{n=1}^{\infty} x^n = x\frac{d}{dx}\left(\frac{1}{1-x} - 1\right) = \frac{x}{(1-x)^2}$.

(d) $\sum_{n=1}^{\infty} \frac{x^n}{n+1} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1} = \frac{1}{x}\sum_{n=1}^{\infty} \int_0^x s^n ds = \frac{1}{x}\sum_{n=1}^{\infty} \int_0^x \left(\frac{1}{1-s} - 1\right) ds = -\frac{\ln(1-x)}{x} - 1.$

(e) The error of approximation by using the first N terms is bounded by $|a_{N+1}| = 1/(N+1)^2$, by the alternating series theorem. So we need $1/(N+1)^2 < 0.0001$, and thus $\boldsymbol{N \geq 100}$.

8. Consider the function $f(x) = \frac{x^2}{1+x^2}$ **with $\boldsymbol{x > 0}$**. Find the point on the graph of f for which the $x-$intercept of the tangent is largest.

Solution: The tangent line at the point $((x_0, f(x_0))$ on the graph has the equation $y = \frac{x_0^2}{1+x_0^2} + (x - x_0)\frac{2x_0}{(1+x_0^2)^2}$. Its $x-$intercept is given by the condition $y = 0$, so

$$x = x(x_0) = \frac{1}{2}x_0 - \frac{1}{2}x_0^3.$$

The maximum of this expression (as a function of $x_0 > 0$) is attained at $x_0 = \sqrt{\frac{1}{3}}$, so the point in question is $\left(\sqrt{\frac{1}{3}}, \frac{1}{4}\right)$.

9. A square $n \times n$ matrix A is called idempotent if $A^2 = A$.

(a) Show: If λ is an eigenvalue of an idempotent matrix A, then $\lambda \in \{0, 1\}$.

(b) Find an example of a 2×2 idempotent matrix whose entries are all nonzero.

(c) Show that any idempotent matrix is diagonalizable.

Solution:

(a) Suppose λ is an eigenvalue with eigenvector v. Then $Av = \lambda\, v$, so $A62v = Av = \lambda\, Av$, or $(1-\lambda)Av = 0$. This implies that either $1-\lambda = 0$, or $Av = 0$. The latter statement is equivalent to $\lambda = 0$.

(b) It's not hard to see that if A is an idempotent matrix, and S an invertible matrix, then SAS^{-1} is idempotent as well, i.e. similar matrices of idempotent matrices are idempotent as well. The matrix $\operatorname{diag}(1,0)$ is idempotent. So for instance, the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$$

is idempotent.

(c) If $A = (v_1, \ldots, v_n)$ are the columns of A, then $A^2 = A$ implies $Av_i = v_i$ for $i = 1, \ldots, n$. So each nonzero column vector of A is an eigenvector of the eigenvalue $\lambda = 1$. Hence the dimension of the eigenspace for $\lambda = 1$ is greater or equal to the rank of A. But $\mathrm{rankA} + \dim \mathrm{KerA} = \mathrm{rankA} + \dim \mathrm{Eig(A, 0)} = n$, and thus the sum of the dimensions of the two eigenspaces equals the dimension of the whole vector space $\mathbb{R}^n$. Taking bases of the two eigenspaces thus yields a linear dependent set with n elements. Thus there is a basis of eigenvectors of A, and hence A is diagonalizable.

10. Consider

$$B = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}.$$

Here $a \in \mathbb{R}$ is a parameter.

(a) For which values of a does B have rank 1?

(b) Find the nth power B^n.

Solution:

(a) The characteristic polynomial is $\det \begin{pmatrix} a-\lambda & 1 \\ 1 & a-\lambda \end{pmatrix} = (a-\lambda)^2 - 1$, giving eigenvalues $\lambda_1 = a-1$ and $\lambda_2 = a+1$. The rank of B is 1 iff its kernel has dimension 1, which is equivalent to exactly one of the eigenvalues being zero. Hence B has rank 1 iff $a = 1$ or $a = -1$.

(b) Diagonalize $B = S \operatorname{diag}(a+1, a-1)\, S^{-1}$; explicitly

$$B = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a+1 & 0 \\ 0 & a-1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Hence

$$B^n = S\,(\operatorname{diag}(a+1, a-1))^n\, S^{-1} = \frac{1}{2} \begin{pmatrix} (a+1)^n + (a-1)^n & (a+1)^n - (a-1)^n \\ (a+1)^n - (a-1)^n & (a+1)^n + (a-1)^n \end{pmatrix}.$$

END OF EXAM