WESTERN WASHINGTON UNIVERSITY – Fall 2006
Graduate Qualifying Exam

Note: *In all problems you must show your work in order to receive credit. Follow the instructions given in each problem. You may use a calculator BUT you must explain thoroughly how you obtained your answers!*

Problem 1. Show that, for an appropriate function $y = f(x)$,

$$\lim_{n\to\infty}\left(\frac{1}{n\sqrt{n^2+1^2}} + \frac{2}{n\sqrt{n^2+2^2}} + \cdots + \frac{n}{n\sqrt{n^2+n^2}}\right) = \int_0^1 f(x)\,dx.$$

Then use this fact to compute the exact value of the limit.

Problem 2. Show that the function $f(x) = x^{1/x}$ is decreasing for all $x \geq e$. Then use this fact to determine which of the two numbers 2006^{2007} and 2007^{2006} is larger.

Problem 3. Use an appropriate substitution to show that

$$\int_{e+\pi}^{e+\pi+\sqrt{\frac{2}{3}}} (x-e-\pi)(x-e-\pi+\sqrt{\frac{2}{3}})(x-e-\pi+2\sqrt{\frac{2}{3}})\,dx = 1.$$

Problem 4. Let (a_n) be a sequence of real numbers. Prove or disprove (by giving a counterexample) each of the following statements:

(a) If the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

(b) If $0 \leq a_n \leq \frac{1}{n}$, then the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is convergent.

Problem 5. The operator $S : \mathbf{R}^3 \to \mathbf{R}^3$ is defined by $S((x_1, x_2, x_3)^T) = (x_2, x_3, x_1)^T$.

(a) Find the matrix A of the operator S if the standard basis $\{e_1, e_2, e_3\}$ is used for $\mathbf{R}^3$, where $e_1 = (1, 0, 0)^T$, $e_2 = (0, 1, 0)^T$, and $e_3 = (0, 0, 1)^T$.

(b) Find the eigenvalues and eigenvectors of A.

Problem 6. The 3×3 matrix A satisfies the equality $AP = PB$, where

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}.$$

Find A^5.

Problem 7. Let $D = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$. Evaluate the double integral

$$\int\int_D \ln(1 + x^2 + y^2)\, dxdy.$$

Problem 8. Let $f(x, y) = (x^2 + y^2)e^{-x-y}$ and let $R = \{(x, y) \in \mathbf{R}^2 : x \geq 0, y \geq 0\}$ denote the first quadrant in the plane.

(a) Find the critical points of f in the interior of the region R and on its boundary (the positive x-axis and positive y-axis).

(b) Find the global maximum and global minimum of f on the region R.

Problem 9. Find the general solution of the system of differential equations

$$\begin{cases} \frac{dx}{dt} &= 4x + y^2 \\ \frac{dy}{dt} &= y \end{cases}.$$

Problem 10. Find the function $y = f(x)$ whose graph is the curve C passing through the point $(2, 1)$ and satisfying the following property: each point (x, y) of C is the midpoint of $L(x, y)$, where $L(x, y)$ denotes the segment of the tangent line to C at (x, y) which lies in the first quadrant.