

# Qualifying Examination

26 March 2012

Explain all your answers and show all your work.

Calculators are **not** permitted.

Time allowed: 3 hours.

1. At noon, a sailboat is 20 km due south of a freighter. The sailboat is traveling due east at 20 km/hr, and the freighter is traveling due south at 40 km/hr. If the visibility at sea is 10 km, could the people on the two ships ever see each other?

2. Let

$$f(x) = \lim_{n \rightarrow \infty} \left\{ \left( \frac{1}{3} + \frac{1}{2 + \left(1 + \frac{x-1}{n}\right)^3} + \frac{1}{2 + \left(1 + \frac{2(x-1)}{n}\right)^3} + \cdots + \frac{1}{2 + \left(1 + \frac{(n-1)(x-1)}{n}\right)^3} \right) \frac{x-1}{n} \right\}.$$

Find the equation of the tangent line to the graph of  $y = f(x)$  at  $x = 1$ .

3. Consider the  $n \times n$  ( $n \geq 2$ ) matrix

$$A = \begin{pmatrix} x & 1 & 1 & \cdots & 1 \\ 1 & x & 1 & \cdots & 1 \\ 1 & 1 & x & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & x \end{pmatrix}.$$

Find all values of  $x$  such that  $A$  is not invertible. For each such  $x$ , determine the rank of  $A$ .

4. Let  $k_i$ ,  $1 \leq i \leq n$ , be distinct real numbers. Show that  $\sum_{i=1}^n a_i e^{k_i t} = 0$  for  $-1 < t < 1$  implies  $a_i = 0$  for all  $1 \leq i \leq n$ .

5. A heated metal plate is placed in the  $xy$ -plane with the temperature at location  $(x, y)$  (in meters) given by  $T(x, y) = -\frac{1}{2}(x^2 + y^2)$  ( $^{\circ}\text{C}$ ). A particle initially at  $(1, 0)$  moves in the plane with velocity  $\vec{v} = \frac{1}{8}(y + (x^2 + y^2)^2 x) \vec{i} + \frac{1}{8}(-x + (x^2 + y^2)^2 y) \vec{j}$  (m/sec). Find the temperature at the location of the particle after one second.

6. Find the points on the surface  $x^2 + xy + y^2 + yz + z^2 = 24$  which are farthest from the  $xy$ -plane.

7. Let  $W$  be a muffin-shaped solid bounded by the sphere  $x^2 + y^2 + z^2 = 8$  from above, the plane  $z = 1$  from below, and the cone  $z = \sqrt{x^2 + y^2}$  from the side. Find the volume of  $W$ .

8. A 1000-gallon tank initially contains a mixture of 450 gallons of cola and 50 gallons of cherry syrup. Cola is added at the rate of 8 gallons per minute, and cherry syrup is added at the rate of 2 gallons per minute. At the same time, a well mixed solution of cherry cola is withdrawn at the rate of 5 gallons per minute. What percentage of the mixture is cherry syrup when the tank is full?

9. Prove the inequality  $\frac{x^2}{4} \leq e^{x-2}$  for all  $x \geq 0$ .

10. Determine whether each of the following series converges or diverges. Explain your reasoning.

(a)  $\sum_{n=1}^{\infty} \sin(1/n^2)$

(b)  $\sum_{n=1}^{\infty} n^{-1-1/n}$