QUALIFYING EXAMINATION

September 16, 2013

You must explain your answers for full credit. Show all work. Write your identifying number on each page you turn in.

Calculators are permitted.

1. For the iterated integral $\int_0^3 \int_0^{2y} f(x, y)\, dxdy$

   (a) Describe the region indicated by the limits of integration using inequalities for x and y. **Sketch** and **label clearly** the region over which the integration is being performed.

   (b) Rewrite the integral with the order of integration reversed.

   (c) Describe the region indicated by the limits of integration using inequalities for the polar coordinate variables r and θ. Rewrite the integral in polar coordinates.

2. A geometry-loving mole has dug a circular tunnel whose central radius is 3 feet centered around a point P in your front yard. The soil displaced above ground is a symmetric ring (like a jello mold or an angelfood cake) whose cross-section directly east of P is the part of the graph of $y = \frac{1}{2} - 8(x-3)^2$ that is above the x-axis. Write and evaluate an integral for the volume of the displaced soil.

3. Consider the recursively defined sequence $\{x_n\}_{n=0}^{\infty}$ where $x_n = \frac{1}{2}x_{n-1}^2 + \frac{1}{2}$ and $x_0 = 0$.

   (a) Compute x_1, x_2, and x_3.

   (b) Show that the sequence $\{x_n\}_{n=0}^{\infty}$ is monotonic (either increasing or decreasing) and bounded.

   (c) Does $\{x_n\}_{n=1}^{\infty}$ converge, and if so to what? Justify your answer.

4. Let $\mathbb{M}$ be the subspace of the vector space $\mathbb{P}_2$ of polynomials of degree at most 2 consisting of all polynomials p such that $p'(0) + p'(3) = 0$. Find a basis for $\mathbb{M}$. Explain carefully how you know that your set of polynomials is a basis.

5. Determine the point or points on the parabola $y = cx^2$ closest to $(0, 1)$ as a function of c for $c > 0$. Justify your answer.

6. Let $f(x) = 10xe^{-2x}$ for $x \geq 0$.
Determine exactly the greatest number a such that f is invertible on $0 \leq x \leq a$.. Sketch the graph of f on this interval, and on the same axes, the graph of its inverse function g. It will be helpful if the graphs are reasonably accurate, so **don't be too hasty**.

   (a) State the domain and range of the inverse function.

   (b) Detrmine $g(1)$ correct to at least **five** decimal places. Explain briefly what you have done.

   (c) Determine $g'(1)$ correct to at least **five** decimal places. Explain what you have done and illustrate with a diagram.

7. Suppose that A is a 4×5 matrix with 4 pivots (that is, rank 4). For each part circle the correct alternative and **explain briefly**.

   (a) Does the equation $A\mathbf{x} = \mathbf{0}$ have a unique solution?

   Yes   No   Could be Either

   (b) Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$?.

   Yes   No   Could be Either

   (c) When the equation $A\mathbf{x} = \mathbf{b}$ is consistent, does it always have a unique solution?

   Yes   No   Could be Either

   (d) State the correct dimensions: the null space of A is a ____ dimensional subspace of $\mathbb{R}^{---}$. The column space of A is a ____ dimensional subspace of $\mathbb{R}^{----}$.

8. The ellipsoid S defined by $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 3$ passes through the point $(-2, 1, -3)$.

   (a) Find a vector normal to S at $(-2, 1, -3)$.

   (b) Find the equation of the plane tangent to S at $(-2, 1, -3)$.

   (c) Find the x-slope and y-slope of the plane in (b).

   (d) In what direction from $(-2, 1, -3)$ should you go in order to reach the surface $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 4$ in the shortest distance? Estimate this distance. Explain your answer briefly, using a diagram.

9. The population $p(t)$ of field mice in the woods behind my house satisfies the logistic equation $\frac{dp}{dt} = p\left(1 - \frac{p}{20}\right)$. (The units are hundreds of mice.)

   (a) Sketch a slope field for the equation. Label any equilibrium solutions and classify them (stable, unstable..) Draw a representative sample of possible solution curves and describe in words all the possible behaviors of solutions as $t \to \infty$.

   (b) Now suppose some feral cats are hunting the mice and catching them at a constant rate, so that the mouse equation is now $\frac{dp}{dt} = p\left(1 - \frac{p}{20}\right) - a$. There is a critical value a_0 of a so that if $a > a_0$, the cats will eradicate the mice, but if $a < a_0$ the mouse population will stabilize at some positive value p_0. Find a_0 and **explain how you know its value**. (No credit for the number a_0 alone, even if correct.)

10. Let $f(x, y) = 2x^2 + x + y^2 - 2$.

   (a) Find the maximum and minimum values of f on the region $x^2 + y^2 \leq 4$. Explain your choices with a diagram.

   (b) Is either the maximum or minimum (or both) from (a) a global maximum or minimum over the entire plane? Explain briefly.