

## Graduate Qualifying Exam Solutions – Fall 2014

**Problem 1.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $f(0) = 2$  and  $|f(x) - f(y)| \leq |x - y|^{5/4}$  for all real numbers  $x$  and  $y$ .

(a) Prove that  $f$  is continuous at all  $x$  using the rigorous  $\epsilon - \delta$  definition of continuity.

(b) Prove that  $f$  is differentiable at all  $x$  using the definition of the derivative.

(c) Compute  $\int_3^6 f(y) dy$ .

**Solution:** For part (a), let  $\epsilon > 0$  and fix  $x$ . We must show that there exists  $\delta > 0$  such that if  $|x - y| < \delta$ , then  $|f(x) - f(y)| < \epsilon$ . Choose  $\delta = \epsilon^{4/5}$ . Then if  $|x - y| < \delta$ ,

$$|f(x) - f(y)| \leq |x - y|^{5/4} < \delta^{5/4} = \epsilon.$$

For part (b), fix  $x$  and let  $h \neq 0$ . Then  $|f(x + h) - f(x)| \leq |h|^{5/4}$ , which implies that

$$0 \leq \frac{|f(x + h) - f(x)|}{|h|} \leq |h|^{1/4}.$$

Taking the limit as  $h \rightarrow 0$ , we see that  $\lim_{h \rightarrow 0} |(f(x + h) - f(x))/h| = 0$  by the squeeze theorem, so that  $\lim_{h \rightarrow 0} (f(x + h) - f(x))/h = 0$ , meaning that  $f'(x)$  exists and is equal to zero.

For part (c),  $f'(x) = 0$  at all  $x$ ,  $f$  is constant, so it must be equal to 2 at all  $x$  (since  $f(0) = 2$ ). Therefore the integral is equal to 6.

**Problem 2.** Let  $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}} dt$ , where  $g(x) = \int_0^{\cos x} (\sin(t^2) + 1) dt$ . Find  $f'(\pi/2)$ .

**Solution:** Using the fundamental theorem of calculus and the chain rule, we have

$$f'(x) = \frac{g'(x)}{\sqrt{1+g(x)^3}} \quad \text{and} \quad g'(x) = -\sin x (\sin(\cos^2 x) + 1).$$

Then  $g'(\pi/2) = -1$ , and  $g(\pi/2) = \int_0^0 \sin(t^2) + 1 dt = 0$ , so  $f'(\pi/2) = -1$ .

**Problem 3.** Find the point on the parabola  $y = 1 - x^2$  in the first quadrant at which the tangent line cuts off the triangle in the first quadrant with smallest area.

**Solution:** Let  $(c, 1 - c^2)$  be a point on the parabola in the first quadrant, hence  $0 \leq c \leq 1$ . The slope of the tangent line to the parabola at this point is  $-2c$ , so the equation of the tangent line is  $y = -2cx + c^2 + 1$ . To determine the side lengths of the triangle this line forms

with the  $x$ - and  $y$ -axes, we find the  $x$ - and  $y$ -intercepts, which are  $(\frac{c^2+1}{2c}, 0)$  and  $(0, c^2 + 1)$ . Thus the area of the triangle in terms of the parameter  $c$  is

$$A(c) = \frac{1}{2}(c^2 + 1) \left( \frac{c^2 + 1}{2c} \right) = \frac{1}{4} \left( c^3 + 2c + \frac{1}{c} \right).$$

To minimize the area, we look for critical points of  $A$  on the given interval and compare the values of  $A$  at these critical points to the value at the endpoints  $c = 0$  and  $c = 1$ . We have  $A' = \frac{1}{4}(3c^2 + 2 - \frac{1}{c^2})$ . Setting  $A' = 0$  and solving the resulting quadratic in  $c^2$  yields  $c^2 = \frac{1}{3}$ , so  $c = \frac{1}{\sqrt{3}}$ . Now  $A(\frac{1}{\sqrt{3}}) = \frac{4}{3\sqrt{3}} < 1 = A(1)$ , and  $A(c) \rightarrow \infty$  as  $c \rightarrow 0^+$ . Therefore the minimum occurs at the point  $(\frac{1}{\sqrt{3}}, \frac{2}{3})$ .

**Problem 4.** Find all points  $P$  on the ellipsoid  $2x^2 + 2y^2 + z^2 = 28$  such that the tangent plane to the ellipsoid at  $P$  is parallel to the plane passing through the three points  $(1, 3, 1)$ ,  $(3, 0, -3)$ , and  $(0, 4, 2)$ .

**Solution:** It's not hard to visualize that there must be two such points. Taking the cross product of two of the displacement vectors between the three points yields the normal vector of the plane passing through them:  $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ . The normal vector to the ellipsoid at a point  $(x, y, z)$  is  $4x\mathbf{i} + 4y\mathbf{j} + 2z\mathbf{k}$ . Therefore, the tangent plane to the ellipsoid at  $P = (x, y, z)$  is parallel to the plane if  $4x\mathbf{i} + 4y\mathbf{j} + 2z\mathbf{k}$  is a scalar multiple of  $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ ; for this to happen there must exist a number  $c$  such that  $4x = c$ ,  $4y = 2c$  and  $2z = -c$ . Therefore  $z = -2x$  and  $y = 2x$ . For  $P$  to be on the ellipsoid, we must then have  $2x^2 + 2(2x)^2 + (-2x)^2 = 14x^2 = 28$ , so  $x = \pm\sqrt{2}$ . So the two points are  $(\sqrt{2}, 2\sqrt{2}, -2\sqrt{2})$  and  $(-\sqrt{2}, -2\sqrt{2}, 2\sqrt{2})$ .

**Problem 5.** Find the volume of the smaller of the two regions enclosed by the surfaces  $z = 1 + x^2 + y^2$  and  $x^2 + y^2 + z^2 = 11$ .

**Solution:** We will set up a triple integral in cylindrical coordinates to calculate the volume which is bounded below by the paraboloid  $z = 1 + x^2 + y^2$  and above by the sphere  $x^2 + y^2 + z^2 = 11$ . To find the intersection of the two surfaces, we solve for  $x^2 + y^2 = z - 1$  in the first equation and substitute in the second equation to get  $z - 1 + z^2 = 11$ . Solving the quadratic yields  $z = 3$  (since  $z > 0$ ), so the curve of intersection in the plane  $z = 3$  is  $x^2 + y^2 = 2$ . This also bounds the shadow of the solid in the  $xy$ -plane, so we use this to determine our bounds of integration. Thus the volume is given by

$$\begin{aligned} \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{1+r^2}^{\sqrt{11-r^2}} r \, dz \, dr \, d\theta &= 2\pi \int_0^{\sqrt{2}} r\sqrt{11-r^2} - r - r^3 \, dr \\ &= 2\pi \left[ -\frac{1}{3}(11-r^2)^{3/2} - \frac{1}{2}r^2 - \frac{1}{4}r^4 \right]_0^{\sqrt{2}} = 2\pi \left( \frac{1}{3}(11)^{3/2} - 11 \right). \end{aligned}$$

**Problem 6.** Show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)} \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz = 2\pi.$$

(Hint: the integral above is improper. To evaluate it, you should compute a triple integral over a suitably chosen bounded region and take the limit as that region grows without bound.)

**Solution:** Since the integrand seems tailor made for spherical coordinates, it makes sense to integrate over a sphere of radius  $r$  centered at the origin, and then take the limit as  $r \rightarrow \infty$ . So we have

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)} \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz &= \lim_{r \rightarrow \infty} \int_0^{2\pi} \int_0^{\pi} \int_0^r \rho e^{-\rho^2} (\rho^2 \sin \phi) \, d\rho \, d\phi \, d\theta \\ &= \lim_{r \rightarrow \infty} 2\pi \left( \int_0^{\pi} \sin \phi \, d\phi \right) \left( \int_0^r \rho^3 e^{-\rho^2} \, d\rho \right). \end{aligned}$$

The first integral is easily shown to equal 2. The second integral can be evaluated by first making the substitution  $w = \rho^2$ , and then using integration by parts. This yields

$$\int_0^r \rho^3 e^{-\rho^2} \, d\rho = \frac{1}{2} [1 - r^2 e^{-r^2} - e^{-r^2}],$$

so we have

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2+z^2)} \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz &= \lim_{r \rightarrow \infty} 2\pi \cdot 2 \cdot \frac{1}{2} [1 - r^2 e^{-r^2} - e^{-r^2}] \\ &= 2\pi. \end{aligned}$$

(Note that l'Hopital's rule can be used to show that  $\lim_{r \rightarrow \infty} r^2 e^{-r^2} = 0$ .)

**Problem 7.** Suppose  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is a linearly independent set of vectors in a vector space  $V$ . Working directly from the definition of linear independence, show that  $\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{w}\}$  is also linearly independent.

**Solution:** Let  $c_1, c_2$ , and  $c_3$  be scalars for the vector space  $V$ , and suppose that

$$c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{v} + \mathbf{w}) + c_3(\mathbf{u} + \mathbf{w}) = \mathbf{0}.$$

Then we have  $(c_1 + c_3)\mathbf{u} + (c_1 + c_2)\mathbf{v} + (c_2 + c_3)\mathbf{w} = \mathbf{0}$ . Since  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is linearly independent, we have  $c_1 + c_3 = 0$ ,  $c_1 + c_2 = 0$ , and  $c_2 + c_3 = 0$ . This is a homogeneous linear system in  $c_1, c_2$  and  $c_3$  with coefficient matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Since the coefficient matrix has full rank, the homogeneous system has only the trivial solution,  $c_1 = c_2 = c_3 = 0$ . Therefore  $\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{w}\}$  is linearly independent.

**Problem 8.** Let  $M$  be the  $1000 \times 1000$  matrix consisting of all 1s. Find the characteristic polynomial for  $M$ .

**Solution:** Let  $p(\lambda)$  be the characteristic polynomial of  $M$ . We will consider the eigenvalues of  $M$ . Since  $M$  has 1000 identical rows, it has rank 1. Thus  $M$  is not invertible, so 0 is an eigenvalue of  $M$ . Its corresponding eigenspace, which is the null space of  $M$ , has dimension 999. The geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity, so we know that  $\lambda^{999}$  divides  $p(\lambda)$ . Now  $p(\lambda)$  has degree 1000, so there is at most one other eigenvalue. We note that the sum of each row of  $M$  is 1000, so if  $\mathbf{1}$  is the vector in  $\mathbb{R}^{1000}$  consisting of all 1s, then  $M\mathbf{1} = 1000\mathbf{1}$ , so 1000 is another eigenvalue of  $M$ . Therefore we must have  $p(\lambda) = \lambda^{999}(\lambda - 1000)$ .

**Problem 9.** Consider the differential equation  $y' = Ay^2$  where  $A$  is a real constant.

- (a) Find the general solution (your solution will contain the parameter  $A$ ).
- (b) Find a value of  $A$  for which there exists a solution  $y(t)$  that satisfies  $y(0) > 0$  and  $y(1) < 0$  and is continuous on an open interval containing the closed interval  $[0, 1]$  or explain why such an  $A$  does not exist.

**Solution:** For (a), if  $A = 0$ , all solutions are constant and all constants are solutions. Note also that for any  $A$ , the constant function  $y = 0$  is a solution. Suppose that  $A \neq 0$ . By the uniqueness theorem, if  $y(t)$  is a solution that equals zero at any time  $t_0$ , then it must be zero for all  $t$ . Thus, if  $y$  is not identically zero, it is never zero and we may use separation of variables to find the general solution. Separating variables give us  $dy/y^2 = A dt$ . Integrating and solving for  $y$  we get  $y(t) = 1/(-At - C)$  where  $C$  is an arbitrary constant.

The situation in (b) is impossible. Any such solution  $y(t)$  would have to equal zero at some  $t \in (0, 1)$  by the intermediate value theorem. However, by the uniqueness theorem, as indicated in the answer to part (a), this implies that  $y(t)$  is identically zero which it can't be if it is to satisfy the given conditions at  $t = 0$  and  $t = 1$ . This can also be seen by using the general solution formula in part (a). If  $y(0) > 0$ , we must have  $C < 0$ . Then  $y(1) = 1/(-A - C)$ . For this to be negative, we must have  $-A < C$ . The solution  $y(t) = 1/(-At - C)$  is undefined at  $t = C/-A$ , but  $C$  and  $-A$  are both negative, so this  $t$  is positive, and since  $-A < C$ ,  $t$  is also less than 1, so  $y(t)$  is not continuous on  $[0, 1]$ .

**Problem 10.** For  $n \in \mathbb{N}$ , define  $a_n = \sqrt{n+1} - \sqrt{n}$ .

- (a) Compute  $\lim_{n \rightarrow \infty} a_n$ .

- (b) Does the series  $\sum_{n=1}^{\infty} (-1)^n a_n$  converge absolutely, converge conditionally or diverge? Justify your answer by using one or more series tests, making sure to explain why the tests apply.

**Solution:** For part (a), the limit is zero. By the mean value theorem applied to  $f(x) = \sqrt{x}$ , we get that  $a_n = 1/(2\sqrt{\xi})$  for some number  $\xi$  between  $n$  and  $n+1$ . Thus  $1/(2\sqrt{n+1}) \leq a_n \leq 1/(2\sqrt{n})$ , so that  $a_n \rightarrow 0$  by the squeeze theorem.

For part (b), the series converges conditionally. The given series converges by the alternating series test: the terms go to zero, alternate in sign, and the inequality in the solution to part (a) shows that  $a_n$  decreases monotonically. The series does not converge absolutely by the comparison test. The inequality in the solution to part (a) shows that  $a_n \geq 1/(2\sqrt{n+1})$ , and the series  $\sum_{n=1}^{\infty} 1/(2\sqrt{n+1})$  diverges (by the integral test or by recognizing it as a  $p$ -series with  $p = 1/2$ , which is known to diverge).