Qualifying Exam

Spring 2011
March 28, 2011 Number ______________

You may use calculators for this exam. Be advised however that every question can be answered without the use of a calculator and more than likely can be answered more efficiently without the use of a calculator.

Problem 1. For which value or values of a are the graphs of $y = ax^2$ and $y = \ln x$ tangent? (This means that the graphs have a common point and the same tangent line at that point.)

Solution. Assume that $x_0 > 0$ is such that $ax_0^2 = \ln x_0$ and that for the corresponding slopes we have $2ax_0 = 1/x_0$. From the last equality, $ax_0^2 = 1/2$, and substituting in the first $1/2 = \ln x_0$. This gives $x_0 = \sqrt{e}$ and $a = 1/(2e)$. It is easy to check that the point $\bigl(\sqrt{e}, 1/2\bigr)$ is on both graphs of $y = \ln x$ and $y = x^2/(2e)$ and that the line $y = x/\sqrt{e} - 1/2$ is the tangent line to both graphs at that point. $\square$

Problem 2. Let $a \in \mathbb{R}$. Consider the function $g_a(x) = \dfrac{a - x}{1 - (1 - a)x}$.

(a) Show that g_a is its own inverse.

(b) Prove that g_a is decreasing on each open interval on which it is defined.

(c) Let $a > 0$. Prove that g_a maps $[0, a]$ onto itself as a bijection.

Solution. (a) Let $x \neq 1/(1 - a)$ and calculate

$$\begin{aligned}
(g_a \circ g_a)(x) &= \frac{a - \frac{a-x}{1-(1-a)x}}{1 - (1-a)\frac{a-x}{1-(1-a)x}} \\
&= \frac{a\bigl(1 - (1-a)x\bigr) - (a - x)}{\bigl(1 - (1-a)x\bigr) - (1-a)(a-x)} \\
&= \frac{-a(1-a)x + x}{1 - (1-a)x - (1-a)a + (1-a)x} \\
&= \frac{\bigl(1 - a(1-a)\bigr)x}{1 - (1-a)a} \\
&= x.
\end{aligned}$$

Thus g_a is its own inverse.

(b) Calculate the derivative

$$g_a'(x) = \frac{-\bigl(1 - (1-a)x\bigr) + (1-a)(a-x)}{\bigl(1 - (1-a)x\bigr)^2} = -\frac{1 - a + a^2}{\bigl(1 - (1-a)x\bigr)^2} = -\frac{\bigl(a - 1/2\bigr)^2 + 3/4}{\bigl(1 - (1-a)x\bigr)^2} < 0.$$

Since the derivative is negative on each open interval on which g_a is defined (b) is proved.

(c) If $a = 1$, then $g_1(x) = 1 - x$. First g_1 is defined on $\mathbb{R}$, so it is defined on $[0, 1]$. Since, $g_1(0) = 1$, $g_1(1) = 0$, and g_1 is decreasing on $(0, 1)$ it maps $[0, 1]$ onto $[0, 1]$. It is a bijection since $g_1 : [0, 1] \to [0, 1]$ is its inverse. If $0 < a < 1$, we have $1/(1 - a) > 1$, so g_a is defined on $[0, a]$. Since, $g_a(0) = a$, $g_a(a) = 0$, and g_a is decreasing on $(0, a)$ it maps $[0, a]$ onto $[0, a]$. It is a bijection since $g_a : [0, a] \to [0, a]$ is its inverse. If $1 < a$, we have $1/(1 - a) < 0$, so g_a is defined on $[0, a]$. Since, $g_a(0) = a$, $g_a(a) = 0$, and g_a is decreasing on $(0, a)$ it maps $[0, a]$ onto $[0, a]$. It is a bijection since $g_a : [0, a] \to [0, a]$ is its inverse. $\square$

Problem 3. Find a 2×2 matrix A with the following three properties: (a) A has real nonzero entries, (b) A is symmetric, (c) the eigenvalues if A are 0 and 1.

Solution. For a matrix to have an eigenvalue 0 it is sufficient to have identical columns. For a matrix to have eigenvalue 1 it is sufficient for the entries in each row to add up to 1. (Then $[1\ 1]^T$ is the eigenvector.) Clearly the matrix $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ satisfies all the properties.

A more thorough solution is as follows. Let $\phi \in (-\pi, \pi]$. Then the vectors $[\cos\phi\ \sin\phi]^T$ and $[-\sin\phi\ \cos\phi]^T$ are mutually orthogonal unit vectors. In fact, each pair of mutually orthogonal unit vectors is in this family. A matrix A from the problem is diagonalizable in the following way

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 0 & -\sin\phi \\ 0 & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} (\sin\phi)^2 & -\frac{1}{2}\sin 2\phi \\ -\frac{1}{2}\sin 2\phi & (\cos\phi)^2 \end{bmatrix}$$

All matrices A required in the problem are given by the above formula. The choice $\phi = -\pi/4$ yields the matrix discovered at the beginning of the proof. $\square$

Problem 4. In this problem we will define two special functions.

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases} \qquad \text{and} \quad \operatorname{Si}(x) = \int_0^x \operatorname{sinc}(t)\, dt.$$

(a) Find the general term of the Maclaurin series of the function Si. The Maclaurin series is the Taylor series about 0.

(b) Calculate $\displaystyle\lim_{x \to 0} \frac{x - \operatorname{Si}(x)}{x^3}$.

Solution. It is well known that the Maclaurin series of the sine function is

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

This series converges for all $x \in \mathbb{R}$. Therefore

$$\operatorname{sinc}(x) = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots$$

This series also converges for all $x \in \mathbb{R}$. Therefore it can be integrated term by term. Hence

$$\operatorname{Si}(x) = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} x^{2n+1} = x - \frac{1}{3 \cdot 3!}x^3 + \frac{1}{5 \cdot 5!}x^5 - \frac{1}{7 \cdot 7!}x^7 + \cdots$$

This series also converges for all $x \in \mathbb{R}$. The last series can be used to find the Maclaurin series of the function whose limit is asked in (b).

$$\frac{x - \operatorname{Si}(x)}{x^3} = \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)!} x^{2n-2} = \frac{1}{3 \cdot 3!} - \frac{1}{5 \cdot 5!}x^2 + \frac{1}{7 \cdot 7!}x^4 - \frac{1}{9 \cdot 9!}x^6 + \cdots$$

This yields $\displaystyle\lim_{x \to 0} \frac{x - \operatorname{Si}(x)}{x^3} = \frac{1}{18}$. $\square$

Problem 5. Let n be a positive integer and let a, b, c, d be real numbers. Consider the $(2n+1) \times (2n+1)$ matrix given on the right and answer the following questions.

(a) Find a necessary and sufficient condition for the given matrix to be invertible.

(b) Assume that the condition for invertibility is satisfied. Give the formula for the inverse.

$$\begin{bmatrix} a & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b \\ 0 & a & \cdots & 0 & 0 & 0 & \cdots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 & b & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & c & 0 & d & \cdots & 0 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c & \cdots & 0 & 0 & 0 & \cdots & d & 0 \\ c & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d \end{bmatrix}$$

Solution. Denote by δ_n the determinant of the given matrix. Then calculate

$$\begin{aligned}
\begin{vmatrix} a & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b \\ 0 & a & \cdots & 0 & 0 & 0 & \cdots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 & b & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & c & 0 & d & \cdots & 0 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c & \cdots & 0 & 0 & 0 & \cdots & d & 0 \\ c & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d \end{vmatrix}
&= a \begin{vmatrix} a & \cdots & 0 & 0 & 0 & \cdots & b & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & \cdots & a & 0 & b & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & c & 0 & d & \cdots & 0 & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c & \cdots & 0 & 0 & 0 & \cdots & d & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d \end{vmatrix}
+ b \begin{vmatrix} 0 & a & \cdots & 0 & 0 & 0 & \cdots & b \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & 0 & \cdots & a & 0 & b & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & c & 0 & d & \cdots & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & c & \cdots & 0 & 0 & 0 & \cdots & d \\ c & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{vmatrix} \\
&= ad \begin{vmatrix} a & \cdots & 0 & 0 & 0 & \cdots & b \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & \cdots & a & 0 & b & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & c & 0 & d & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c & \cdots & 0 & 0 & 0 & \cdots & d \end{vmatrix}
- bc \begin{vmatrix} a & \cdots & 0 & 0 & 0 & \cdots & b \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & \cdots & a & 0 & b & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & c & 0 & d & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c & \cdots & 0 & 0 & 0 & \cdots & d \end{vmatrix} \\
&= (ad - bc) \begin{vmatrix} a & \cdots & 0 & 0 & 0 & \cdots & b \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & \cdots & a & 0 & b & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & c & 0 & d & \cdots & 0 \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c & \cdots & 0 & 0 & 0 & \cdots & d \end{vmatrix}
\end{aligned}$$

Thus, $\delta_n = (ad - bc)\delta_{n-1}$. Continuing recursively we get

$$\delta_n = (ad - bc)\delta_{n-1} = (ad - bc)^2 \delta_{n-2} = \cdots = (ad - bc)^{n-1}\delta_1 = (ad - bc)^n,$$

since it is an easy calculation to show that

$$\delta_1 = \begin{vmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{vmatrix} = 1 \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Hence, the given matrix is invertible if and only if $ad - bc \neq 0$. Recall that for 2×2 matrix we have

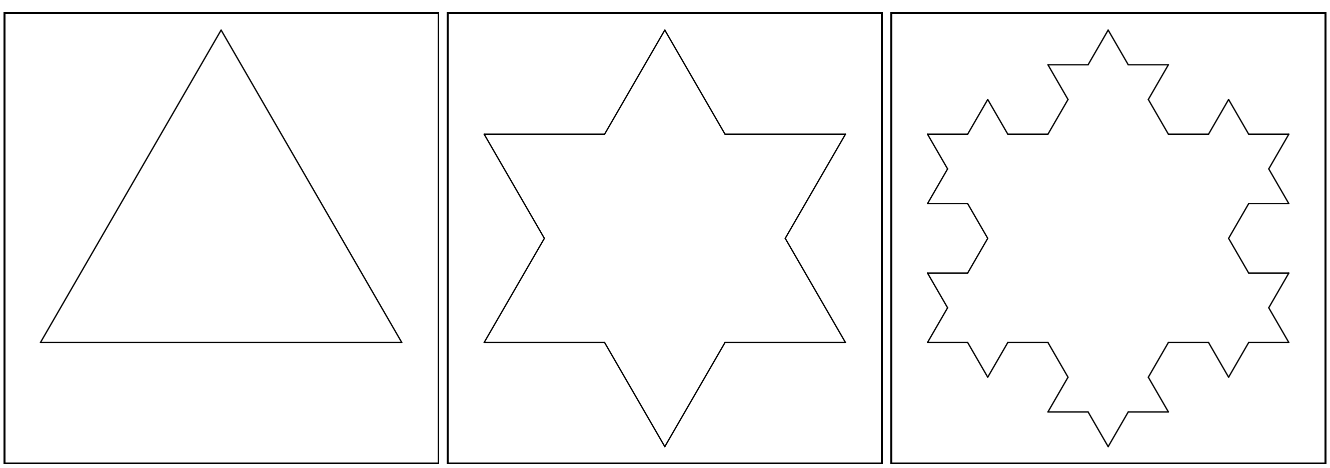
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

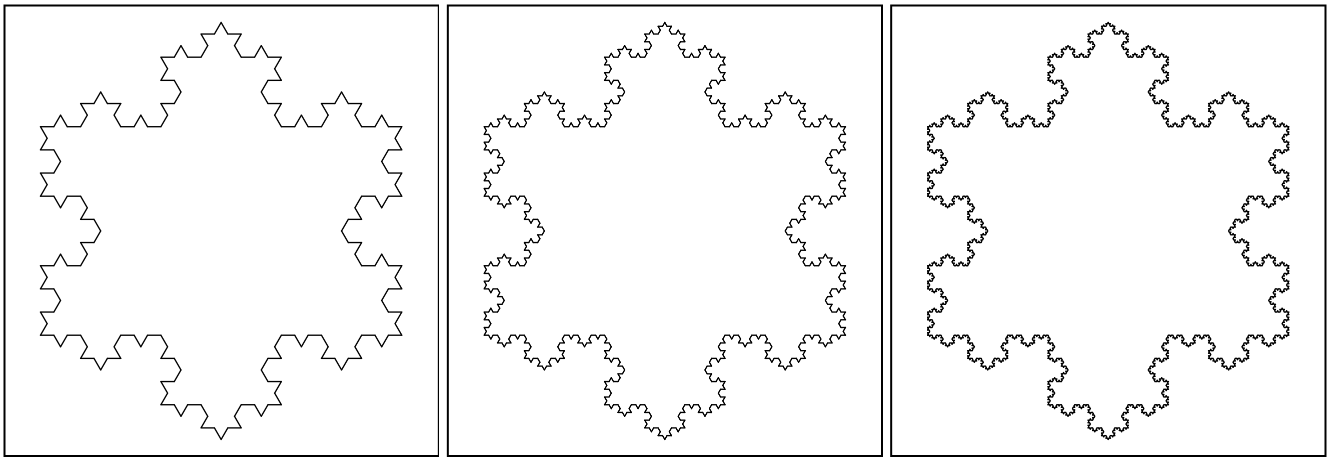
Similarly we have

$$\begin{bmatrix} a & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b \\ 0 & a & \cdots & 0 & 0 & 0 & \cdots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 & b & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & c & 0 & d & \cdots & 0 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c & \cdots & 0 & 0 & 0 & \cdots & d & 0 \\ c & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d \end{bmatrix} \begin{bmatrix} d & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -b \\ 0 & d & \cdots & 0 & 0 & 0 & \cdots & -b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 0 & \cdots & d & 0 & -b & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -c & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -c & \cdots & 0 & 0 & 0 & \cdots & a & 0 \\ -c & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a \end{bmatrix}$$

equals the $(2n+1) \times (2n+1)$ diagonal matrix with the first and the last n diagonal entries equal $ad - bc$ and with 1 in the middle. Thus the inverse of the given matrix is the matrix of the same form in which a is replaced by $d/(ad - bc)$, b is replaced by $-b/(ad - bc)$, c is replaced by $-c/(ad - bc)$ and d is replaced by $a/(ad - bc)$. $\square$

Problem 6. The Koch snowflake is a figure that is constructed in the following way: The Koch snowflake at step 0, denoted by K_0, is an equilateral triangle with sides of length 1. Then, on each step we break every side of the figure into three equal segments, on every middle segment we build an equilateral triangle (facing outwards from our figure) and finally we throw the middle segments away. The Koch snowflakes $K_0, K_1, K_2, K_3, K_4, K_5$ are given in the figures below. Calculate the area A_n enclosed by the Koch snowflake K_n and determine the limit of A_n as $n \to +\infty$.





Solution. To calculate the area we need to count the number of sides of each snowflake. K_0 has 3 sides, K_1 has $3 \cdot 4$ sides, K_2 has $3 \cdot 4 \cdot 4$ sides, so K_n has $3 \cdot 4^n$ sides. Next calculate the length of the sides. The sides of K_0 have length 1, the sides of K_1 have length $1/3$ sides, the sides of K_2 have length $1/9$ sides, so the sides of K_n have length $(1/3)^n$ sides. The area A_0 is $\sqrt{3}/4$. To get A_1 we add three (the number of sides of K_0) equilateral triangles with sides $1/9$ (the length of sides of K_1). Thus

$$
\begin{aligned}
A_1 &= A_0 + 3\frac{\sqrt{3}}{4}\left(\frac{1}{3}\right)^2 = A_0 + \frac{\sqrt{3}}{12} \\
A_2 &= A_1 + 3 \cdot 4\frac{\sqrt{3}}{4}\left(\frac{1}{9}\right)^2 = A_1 + \frac{\sqrt{3}}{12}\,\frac{4}{9} \\
&\vdots \\
A_n &= A_{n-1} + 3 \cdot 4^{n-1}\frac{\sqrt{3}}{4}\left(\frac{1}{3^n}\right)^2 = A_{n-1} + \frac{3\sqrt{3}}{4 \cdot 9}\left(\frac{4}{9}\right)^{n-1} = A_{n-1} + \frac{\sqrt{3}}{12}\left(\frac{4}{9}\right)^{n-1}
\end{aligned}
$$

Thus

$$
\begin{aligned}
A_n &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{12}\,\frac{4}{9} + \frac{\sqrt{3}}{12}\left(\frac{4}{9}\right)^2 + \cdots + \frac{\sqrt{3}}{12}\left(\frac{4}{9}\right)^{n-1} \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\,\frac{1-\left(\frac{4}{9}\right)^n}{1-\left(\frac{4}{9}\right)} \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}\,\frac{9}{5}\left(1-\left(\frac{4}{9}\right)^n\right) \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20}\left(1-\left(\frac{4}{9}\right)^n\right) \\
&= \frac{2\sqrt{3}}{5} - \frac{3\sqrt{3}}{20}\left(\frac{4}{9}\right)^n
\end{aligned}
$$

Hence $\lim_{n\to+\infty} A_n = \dfrac{2\sqrt{3}}{5}$. $\square$

Problem 7. This is a mathematical loan problem. Assume that the interest on this loan is compounded continuously and the payments are made continuously. This assumption allows us to use differential equations to model this loan. The annual interest rate on this loan is 6%. (This is $0.5\% = \frac{1}{200}$ monthly rate.) The payments are made continuously with the monthly rate of $1000\,\dfrac{e^{3/2}}{e^{3/2}-1}$ dollars. (This is approximately \$1,287.22 dollars monthly payment.) The current amount of this loan is \$200,000 dollars.

(a) Set up an initial value problem which models this loan.

(b) Solve the stated initial value problem.

(c) How long will it take for this loan to be paid off? Please answer in years rather than months.

Solution. Denote by $L(t)$ the amount of loan at time t. Then the initial value problem is

$$L'(t) = \frac{1}{200}L(t) - 1000\frac{e^{3/2}}{e^{3/2}-1}, \quad L(0) = 200000.$$

It is easier to solve this problem in the general case with r being the interest rate, p the monthly payment and L_0 the initial loan amount.

$$L'(t) = rL(t) - p, \quad L(0) = L_0.$$

A standard way to solve a linear equation with constant coefficients is as follows:

$$\begin{aligned}
L'(t) - rL(t) &= -p \\
e^{-rt}L'(t) - re^{-rt}L(t) &= -e^{-rt}p \\
\frac{d}{dt}\left(e^{-rt}L(t)\right) &= -e^{-rt}p \\
e^{-rt}L(t) &= e^{-rt}\frac{p}{r} + C \\
e^{-rt}L(t) &= e^{-rt}\frac{p}{r} - \frac{p}{r} + L_0 \\
L(t) &= \frac{p}{r} + \left(L_0 - \frac{p}{r}\right)e^{rt} \\
L(t) &= \frac{p}{r} + \frac{1}{r}\left(rL_0 - p\right)e^{rt}
\end{aligned}$$

Hence the solution is

$$L(t) = 200000\frac{e^{3/2}}{e^{3/2}-1} + 200\left(1000 - 1000\frac{e^{3/2}}{e^{3/2}-1}\right)e^{t/200} = \frac{200000}{e^{3/2}-1}\left(e^{3/2} - e^{t/200}\right)$$

To answer (c) we set $L(t) = 0$ and solve for t. This is almost obvious for the last expression. We proceed to solve the general equation:

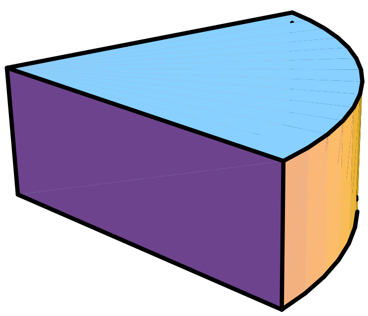
$$\begin{aligned}
\frac{p}{r} + \left(L_0 - \frac{p}{r}\right)e^{rt} &= 0 \\
e^{rt} &= \frac{p}{p - rL_0} \\
t &= \frac{1}{r}\ln\left(\frac{p}{p - rL_0}\right) \\
t &= 200\ln\left(\frac{1000\frac{e^{3/2}}{e^{3/2}-1}}{1000\frac{e^{3/2}}{e^{3/2}-1} - 1000}\right) \\
t &= 200\ln\left(\frac{e^{3/2}}{e^{3/2} - \left(e^{3/2}-1\right)}\right)
\end{aligned}$$

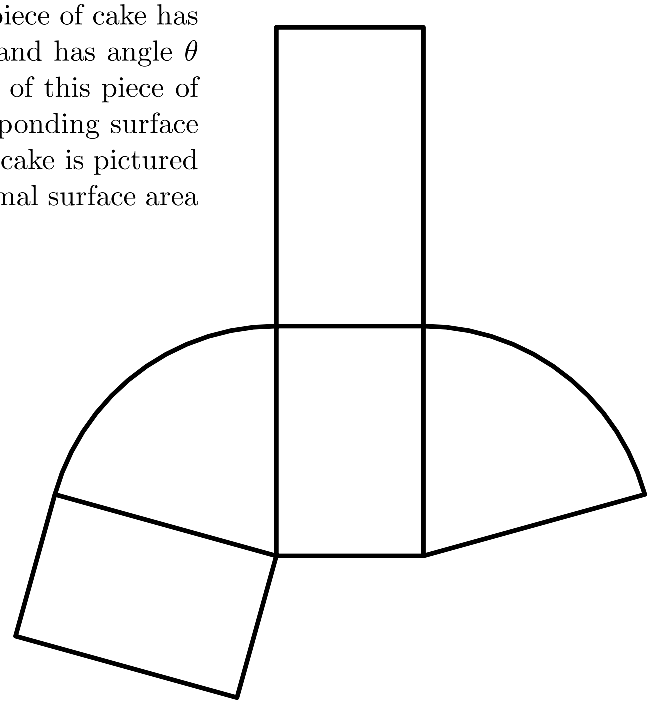
$$t = 200 \ln\left(e^{3/2}\right)$$
$$t = 300.$$

Thus, it will take 25 years to pay off this loan.

Remark. Notice that $\lim_{r\to 0} \frac{1}{r} \ln\left(\frac{p}{p - rL_0}\right)$ is an interesting limit. Since the meaning (see the third line in the preceding calculations) of the expression whose limit we are seeking is the time to pay of the loan of L_0 with payments of p, the answer with 0 interest should clearly be L_0/p. To get this answer without knowing the context of the formula is a nice exercise in the application of l'Hôpital's rule. □

Problem 8. A piece of cake is shown below. This piece of cake has height z, is cut from a cylindrical cake of radius r and has angle θ at the center of the piece. Assume that the volume of this piece of cake is 1. Calculate z, r and θ for which the corresponding surface area of the cake is minimal. The surface area of the cake is pictured to the right. **Note:** You can <u>assume</u> that such minimal surface area exists. You do not need to prove that.





Solution. The volume of the cake is $zr^2\theta/2 = 1$. Thus $z = 2/(r^2\theta)$. The surface area is

$$S = 2zr + r^2\theta + zr\theta = \frac{4}{r\theta} + r^2\theta + \frac{2}{r}.$$

To find the critical points we solve the system of equations:

$$\frac{\partial S}{\partial r} = -\frac{4}{r^2\theta} + 2r\theta - \frac{2}{r^2} = \frac{-4 + 2r^3\theta^2 - 2\theta}{r^2\theta} = 0$$
$$\frac{\partial S}{\partial \theta} = -\frac{4}{r\theta^2} + r^2 = \frac{-4 + r^3\theta^2}{r\theta^2} = 0$$

From the second equation we have $r^3\theta^2 = 4$. Substituting this into the first equation we get $2\theta = 4$. Thus, $\theta = 2, r = 1$ is the only critical point. To show that this is a local minimum we use the second partial derivative test:

$$\frac{\partial^2 S}{\partial r \partial r} = \frac{8}{r^3\theta} + 2\theta - \frac{4}{r^3}, \qquad \frac{\partial^2 S}{\partial r \partial r}(1,2) = 4 > 0$$
$$\frac{\partial S}{\partial \theta \partial r} = \frac{4}{r^2\theta^2} + 2r, \qquad \frac{\partial S}{\partial \theta \partial r}(1,2) = 3$$
$$\frac{\partial S}{\partial \theta \partial \theta} = \frac{8}{r\theta^3} \qquad \frac{\partial S}{\partial \theta \partial \theta}(1,2) = 1$$

Now calculate
$$\frac{\partial^2 S}{\partial r \partial r}(1,2)\,\frac{\partial S}{\partial \theta \partial \theta}(1,2) - \left(\frac{\partial S}{\partial \theta \partial r}(1,2)\right)^2 = 4\cdot 1 - 9 = -5 < 0.$$
Thus the point $r = 1, \theta = 2$ is a local minimum of the function $S(r,\theta)$. The corresponding minimal area is $S(1,2) = 6$. Since we assume that the global minimum area exists and since we found only one local minimum this local minimum must be global. Hence the optimal piece of cake with $r = 1, \theta = 2, z = 1$ has volume 1 and the minimal area 6.

Remark. The following part of the solution was not expected on the exam. It is included here for completeness. To show that this is actually the global minimum is little tricky. A "simple" way to do it is to rewrite the formula for $S(r,\theta)$ as a sum of nonnegative quantities. To find such a formula we used the fact that we expect the local minimum of 6 at $(1,2)$ to be the global minimum. Here is a formula that proves that the value 6 at $(1,2)$ is the global minimum:

$$S(r,\theta) = \frac{4}{r\theta} + r^2\theta + \frac{2}{r} = \left(\frac{2}{\sqrt{r\theta}} - r\sqrt{\theta}\right)^2 + 2\left(\frac{2}{\sqrt{r}} + \frac{1}{r}\right)\left(1 - \sqrt{r}\right)^2 + 6$$

This formula can be verified by a simplification of its left hand side.

This problem can be solved using the Lagrange multiplier method. But solving four nonlinear equations with four unknowns is tricky. Here is how it goes. We need to minimize

$$r\theta z + r^2\theta + 2rz$$

under the constraint
$$\frac{r^2\theta z}{2} = 1.$$

The corresponding Lagrange equations are

$$\begin{aligned} \theta z + 2r\theta + 2z &= \lambda r\theta z \\ rz + r^2 &= \lambda r^2 z/2 \\ r\theta + 2r &= \lambda r^2\theta/2 \\ r^2\theta z &= 2 \end{aligned}$$

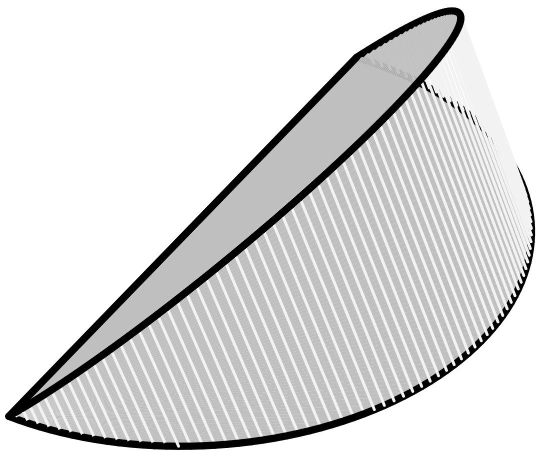
First solve the second and third equation for z and θ, respectively,

$$z = \frac{2r}{\lambda r - 2} \qquad \text{and} \qquad \theta = \frac{4}{\lambda r - 2}.$$

Then substitute in the first and fourth equation and simplify. Amazingly, the first equation is a quadratic equation in λr, while the fourth equations is quite simple:

$$\begin{aligned} (\lambda r)^2 - 4\lambda r &= 0 \\ 4r^3 &= (\lambda r - 2)^2. \end{aligned}$$

Hence $\lambda r = 4$ and $r = 1$. Thus $\lambda = 4$ and $\theta = 2$, $z = 1$. Since we found only one point satisfying the Lagrange conditions and since we are given that the surface area has the minimum value under the given constraint, we conclude that the minimal surface area must be 6. □

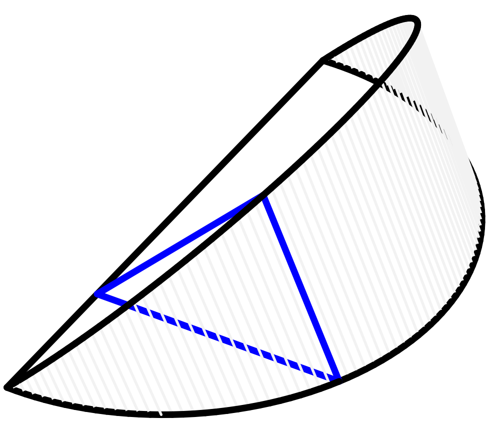


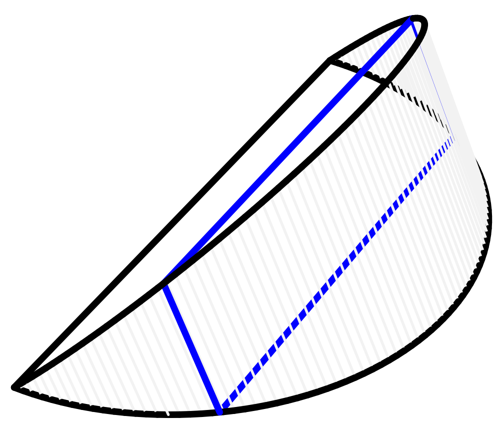
Problem 9. The picture on the left shows a simplified taco. It is a flat unit disk tortilla which is folded along its diameter so that the flat semi-disks planes form an angle of $\pi/4$. Determine the volume of the filling that can be fitted in this taco if the filling has to be lined up with the edges of the semi-disks, as shown in the figure to the left.

Solution. We can calculate the volume for any angle $\alpha \in (0, \pi)$ instead of the special case $\alpha = \pi/4$. We will slice the taco with planes orthogonal to the diameter along which the taco is folded, see the picture on the left below. If we cut at the distance x from the center, then the intersection will be an isosceles triangle with the equal sides of length $\sqrt{1-x^2}$ and the angle between them α. The area of this triangle is $\frac{1}{2}\big(1-x^2\big)\sin\alpha$. Integrating this quantity over $-1 \le x \le 1$ we get the volume:

$$\frac{\sin\alpha}{2}\int_{-1}^{1}(1-x^2)\,dx = \frac{\sin\alpha}{2}\left(2-\frac{2}{3}\right) = \frac{2}{3}\sin\alpha.$$

In the special case $\alpha = \pi/4$ we get $\sqrt{2}/3$.





An alternative way to calculate the volume is to slice by planes that are parallel to the diameter along which the taco is folded (the sharp edge of the taco) and contain the white line segments in the picture, see the picture on the right above. These white line segments bound the filling of the taco. Then the integration parameter $l \in \big[0, \cos(\alpha/2)\big]$ and the cross sections are rectangles. Using the appropriate right-triangles and trigonometry we calculate that the sides of the rectangle at the distance l from the sharp edge of the taco are:

$$2\,l\,\tan(\alpha/2)\ \boxed{\begin{array}{l}\text{this is the white line seg-}\\ \text{ment at distance } l \text{ from the}\\ \text{sharp edge of the taco}\end{array}} \qquad \text{and} \qquad 2\sqrt{1-\left(\frac{l}{\cos(\alpha/2)}\right)^2}\ \boxed{\begin{array}{l}\text{this is the corre-}\\ \text{sponding chord of}\\ \text{the unit disk}\end{array}}.$$

Then the volume is given by the integral

$$\begin{aligned}\int_0^{\cos(\alpha/2)} 2\,l\,\tan(\alpha/2)\,2\sqrt{1-\left(\frac{l}{\cos(\alpha/2)}\right)^2}\,dl &= \left|\begin{array}{rcl} t &=& 1-\left(\frac{l}{\cos(\alpha/2)}\right)^2 \\ \left(\cos(\alpha/2)\right)^2 dt &=& -2l\,dl \end{array}\right| \\ &= 2\left(\cos(\alpha/2)\right)^2\,\tan(\alpha/2)\int_0^1 \sqrt{t}\,dt \\ &= 2\cos(\alpha/2)\sin(\alpha/2)\frac{2}{3}\,t^{3/2}\Big|_0^1 \\ &= \frac{2}{3}\sin(\alpha)\end{aligned}$$

□

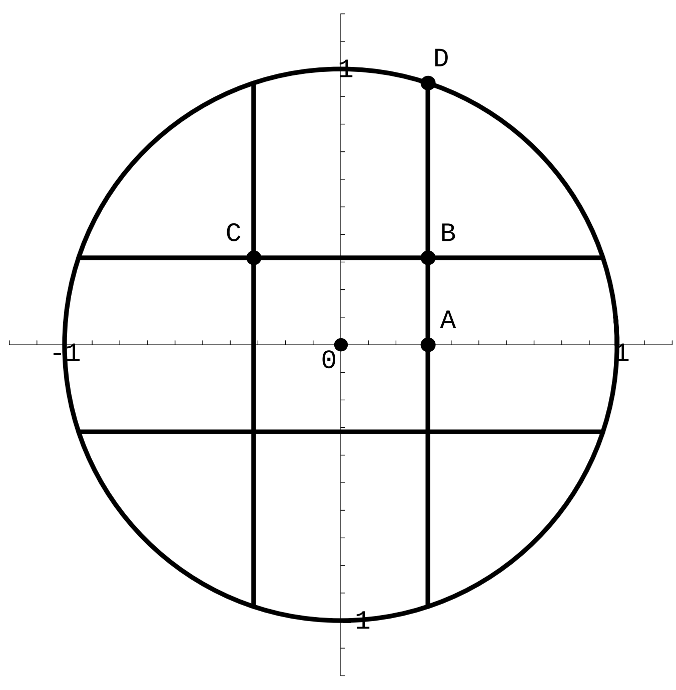
Problem 10. A unit disk is divided in nine parts by two pairs of parallel lines. The lines are at the same distance from the center and the pairs of parallel lines are mutually orthogonal. See the figure on the right. Is it possible to choose the lines in such a way that all nine parts have identical areas? Justify your answer with exact calculations.

Solution. This is not possible. The proof is by contradiction. Assume that all nine areas are equal. Since the area of the unit disk is π, the area of each piece is $\pi/9$. Place the coordinate system at the center of the disk and with the coordinate axes parallel to the given lines. Let A, B, C, D be points as given on the picture below. Since the middle piece is a square, its side is $\sqrt{\pi}/3$. Therefore the distance of the parallel lines from the center is $\sqrt{\pi}/6$. The coordinates of the labeled points are

$$A = \left(\frac{\sqrt{\pi}}{6}, 0\right), \quad B = \left(\frac{\sqrt{\pi}}{6}, \frac{\sqrt{\pi}}{6}\right), \quad C = \left(-\frac{\sqrt{\pi}}{6}, \frac{\sqrt{\pi}}{6}\right), \quad D = \left(\frac{\sqrt{\pi}}{6}, \sqrt{1-\frac{\pi}{36}}\right).$$

The area immediately above the square is $\pi/9$. Since the length of its base CB is $\sqrt{\pi}/3$ and since its top is concave down, the length of the vertical line segment BD must be less than $\sqrt{\pi}/3$. In fact, the length of this vertical straight edge must is

$$\sqrt{1-\frac{\pi}{36}} - \frac{\sqrt{\pi}}{6}.$$



As a consequence we have the following inequalities:

$$\begin{aligned}\sqrt{1-\frac{\pi}{36}} - \frac{\sqrt{\pi}}{6} &< \frac{\sqrt{\pi}}{3} \\ \sqrt{1-\frac{\pi}{36}} &< \frac{\sqrt{\pi}}{2} \\ 1-\frac{\pi}{36} &< \frac{\pi}{4} \\ 1 &< \frac{10}{36}\pi \\ \frac{36}{10} &< \pi.\end{aligned}$$

Contradiction! □