Spring 2010 Qualifying Exam
March 29, 2010

Directions: Explain all your answers and show all your work. No calculators.

1. Let B denote the matrix $\begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$.

   (a) Find an invertible 2×2-matrix P with real entries such that $P^{-1}BP$ is diagonal.

   (b) Compute B^{1024}. Write your answer in terms of $a = 2^{1024}$.

   (c) Is there a 2×2-matrix Q with real entries such that the matrix $Q^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}Q$ is diagonal?

2. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

3. (a) Determine all values of p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges and justify your conclusion.

   (b) Find all values of x for which the following series converges and all values of x for which it diverges, and justify your conclusions. Be specific about your answers, and remember to consider all x.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}(3x-1)^n$$

4. By setting up and computing a multiple integral find the volume of the solid bounded by the surfaces $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$.

5. Consider the initial value problem $y'(t) = ty^\alpha$ and $y(0) = 1$, where α is a real parameter.

   (a) Solve this problem when $\alpha = 1$.

   (b) Find the solution $y_\alpha(t)$ for all $\alpha \neq 1$. Note that the solution will depend on α.

   (c) For each α, determine the largest interval for which the solution $y_\alpha(t)$ is defined.

6. Let V be a finite dimensional vector space of dimension $n \geq 1$ and let $f\colon V \to V$ be a linear map with kernel $\mathrm{Ker}(f)$ and image $\mathrm{Im}(f)$. Prove that the following are equivalent:

   (a) $\mathrm{Im}(f) = \mathrm{Ker}(f)$

   (b) $f \neq 0$, $f^2 = 0$, n is even, and the rank of f is $\frac{n}{2}$.

7. Find the maximum and minimum values of $f(x, y) = xy + y$ subject to the constraint $x^2 + y^2 = 1$.

8. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}$$

for all $x > 0$.

9. A person on the shore of a circular lake with radius 2 miles wants to arrive at the point diametrically opposite on the other side of the lake in the shortest possible time. If the person can walk at the rate of 4 miles/hour and row a boat at 2 miles/hour, how should they proceed?

10. A large, well-stirred tank initially contains 100 gallons of clean water. Water is drained from the tank at a (variable) rate of $r(t)$ gallons per minute. Water polluted with a particular pesticide enters the tank from two different sources. The concentration of pesticide in the water entering the tank from each source varies with time (and may be different for each source), but the inflow rate of water from each source remains constant (at r_1 gallons per minute for the first source and r_2 gallons per minute for the second). Let $y(t)$ equal the amount of the pesticide in the tank at time t, and $c_i(t)$ be the concentration of the pesticide in the water entering the tank from source i, for $i = 1, 2$. Write a differential equation and initial condition for $y(t)$ that describes this situation. DO NOT ATTEMPT TO SOLVE FOR $y(t)$. Hint to get started: Find an expression for the amount of liquid in the tank at any time t.