

Qualifying Examination

12 September 2016

Explain all your answers and show all your work.

Calculators are **not** permitted.

Time allowed: 3 hours

1. A straight road follows a diagonal of a square city of side 10 miles. The density of the population at distance x miles from the road is well-approximated by $\rho(x) = 15 - 3x/\sqrt{2}$ thousand people per square mile. Find the total population of the city.

2. Consider the function

$$f(x) = \arcsin \sqrt{x} - \arctan \left(\sqrt{\frac{x}{1-x}} \right).$$

(a) What is the domain of f ?

(b) Calculate the derivative of f .

(c) Sketch the graph of f .

3. A small triangular island occupies the region bounded by the y -axis, the line $y = -2$, and the line $y = 2 - x$. The temperature at each point (x, y) on the island is given by $T(x, y) = 60 + x^2 + xy + 2y^2 - 3x + 2y$. Find the distance between the hottest and coldest points on the island.

4. A wooden object is a ball in \mathbb{R}^3 of radius R , centered at $(0, 0, 0)$, with the part lying in the infinite cylinder $\{(x, y, z) : x^2 + y^2 < r^2\}$ removed, where $r < R$. (In other words, a hole of radius r has been drilled through the center of the ball to make the object). Find the volume $V(R, r)$ of the object in terms of R and r .

5. A solid tetrahedron T has vertices $(3, 4, 5)$, $(3, 4, 6)$, $(4, 5, 5)$ and $(4, 6, 5)$.

(a) Find the volume of T .

(b) Find the surface area of T .

6. Let $x_0 \in (0, \frac{\pi}{2})$. Define the sequence $\{x_n\}$ by $x_n = \sin x_{n-1}$ for $n \geq 1$.

(a) Prove rigorously that $\lim_{n \rightarrow \infty} x_n = 0$.

(b) Find the limit $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n}$. Justify your answer.

7. Determine whether the integral

$$\int_1^{\infty} \frac{9x^2 + 11}{\sqrt{x^5 + 14x^4 - 2\cos(x^3)}} dx$$

converges or diverges. Give a careful and complete justification for your answer, stating any theorems you use, and verifying that their hypotheses are satisfied when you apply them.

8. Consider the family of matrices

$$A_x = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

For which values of x can A_x be diagonalized? Justify your answer.

9. Find all real 2×2 matrices A satisfying $A^2 = I$. Give your answer as an explicit formula.

10. A large and prosperous country is surrounded by a giant wall. Its majestic demagogue has decreed that its growing population $P(t)$ must never exceed 400,000,000. Accordingly, the rate of growth of the population at any time is proportional to the product of the population at that time and the additional population permitted by the demagogue at that time.

- (a) Write down a differential equation for $P(t)$.
- (b) One day, the demagogue's demographer reports that $P(t)$ is passing through a point of inflection, for which he is immediately fired. Calculate the population of the country when this happens.