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# Graduate Qualifying Exam – Spring 2014 Solutions

**Problem 1.** Suppose $n$ is a fixed positive integer larger than 1. Let $f(x) = x^n e^{-x}$ restricted to $x > 0$.

(a) Does $f$ have any horizontal asymptotes? If yes, compute them.
(b) Does $f$ have a global maximum? If yes, find it.
(c) Find the inflection points of $f$.
(d) Use (a) to prove that $ex \leq ne^{x/n}$ for all $x > 0$.

(1) We compute $\lim_{x\to\infty} x^n e^{-x}$ using L'Hopitals rule repeatedly to find that it equals $\lim_{x\to\infty} n!e^{-x} = 0$.

(2) One finds $f'(x) = x^{n-1}e^{-x}(n-x)$. Setting this equal to 0 yields the only critical point $x = n$. Using the first derivative test, one checks this is a local maximum. In addition, since the derivative is positive before $x = n$ and negative after this point, this local maximum is a global maximum.

(3) To compute inflection points, we compute the second derivative:

$$\begin{aligned} f''(x) &= n((n-1)x^{n-2}e^{-x} + x^{n-1}(-e^{-x})) - (nx^{n-1}e^{-x} - x^n e^{-x}) \\ &= x^{n-2}e^{-x}(n(n-1) - nx - nx + x^2). \end{aligned}$$

This always exists on the domain, and vanishes exactly when $x^2 - 2nx + n(n-1) = 0$. Using the quadratic formula, this occurs when $x = n \pm \sqrt{n}$. Finally, we compute the sign of $f''(x)$ on either side of these values. In particular, since $f''(x) = x^{n-2}e^{-x}(x-(n+\sqrt{n}))(x-(n-\sqrt{n}))$, we find that $f''(x) > 0$ on $(0, n-\sqrt{n})$, $f''(x) < 0$ on $(n-\sqrt{n}, n+\sqrt{n})$ and $f''(x) > 0$ on $(n+\sqrt{n}, \infty)$. Therefore, $x = n \pm \sqrt{n}$ are the inflection points of $f(x)$.

(4) By part (a), $x^n e^{-x} \leq n^n e^{-n}$ for all $x > 0$. Thus, $x^n \leq n^n e^{x-n}$. Taking $n$th roots of both sides, we get $x \leq ne^{x/n}e^{-1}$ whence the result.

**Problem 2.** A person 6ft tall walks 5ft/sec along one edge of a straight road 30ft wide. On the other edge of the road, ahead of the person, there is a light atop a pole 18ft high. How fast is the length of the person's shadow increasing when the person is 40ft from the point directly across the road from the pole?

Let $x$ denote the distance between the person and the point of minimal distance between the persons side of the road and the pole. Let $y$ denote the length of the person's shadow. We must compute $\frac{dy}{dt}|_{x=40}$. To this end we note that there is a right triangle whose height is 18ft and whose base is $y + \sqrt{30^2 + x^2}$. Furthermore, this right triangle contains the right triangle with height 6ft and base $y$. Therefore, using similar triangles, we deduce that

$$\frac{y + \sqrt{30^2 + x^2}}{18} = \frac{y}{6}$$

which implies that $y = \frac{1}{2}\sqrt{30^2 + x^2}$. Therefore, $\frac{dy}{dt} = \frac{1}{4}(30^2 + x^2)^{-1/2} 2x \frac{dx}{dt}$, and so $\frac{dy}{dt}|_{x=40} = 2ft/s$.

**Problem 3.**

(1) Evaluate $\int_0^1 e^{\sqrt{x}} dx$.

(2) Find a function $f$ such that

$$x^2 = 1 + \int_1^{2x} \sqrt{1+[f(t)]^2}dt$$

for all $x > 2$.

(1) We first use substitution: $u = \sqrt{x}$ implies that $du = \frac{1}{2\sqrt{x}}dx$ which, in turn, implies that $2udu = dx$. Therefore, the integral equals

$$\int_0^1 2ue^u.$$

We compute this using integration by parts: $\int 2ue^u = 2(ue^u - \int e^u du) = 2(u-1)e^u$. This evaluates to 2.

(2) Taking derivatives of both sides with respect to $x$ and using the Fundamental Theorem of Calculus yields the constraint

$$x = \sqrt{1+f(2x)^2}.$$

This implies that $f(2x)^2 = x^2 - 1$ so that $f(x)^2 = (\frac{x}{2})^2 - 1$. Therefore, $f(x) = \sqrt{(\frac{x}{2})^2 - 1}$ is a solution.

**Problem 4.** Consider the region $R$ in the $xy$-plane bounded by $y^2 = 2(x-3)$ and $y^2 = x$. Find the volume of the solid generated by rotating $R$ around the $x$-axis.

By graphing and setting the curves equal to each other, one deduces that the relevant volume is given by the expression

$$\int_0^6 \pi x dx - \int_3^6 \pi 2(x-3)dx = 18\pi - 9\pi = 9\pi.$$

**Problem 5.** Find the maximal area of a rectangle that is inscribed in the ellipse $(x/2)^2 + y^2 = 1$ and whose sides are parallel to the coordinate axes in the $xy$-plane.

Let $(x, y)$ be the intersection of the rectangle with the ellipse in the first quadrant. We need to maximize $f(x, y) = 4xy$ subject to $g(x, y) = x^2 + 4y^2 - 4 = 0$. By the method of Lagrange multipliers, a maximum occurs as a solution to $4y = 2\lambda x$ and $4x = 8\lambda y$, which implies that

$$\frac{2y}{x} = \lambda = \frac{x}{2y}.$$

Therefore, at the maximum, $x^2 = 4y^2$. Given the constraint, we deduce that $x^2 = 4y^2 = 2$. The fact that $(x, y)$ is in the first quadrant implies that $x = \sqrt{2}$ and $y = \frac{1}{\sqrt{2}}$ for a maximum area of 4 units squared.

**Problem 6.** Let $\vec{u} \in \mathbb{R}^n$ be such that $\vec{u}^T\vec{u} = 1$. Define the matrix $M := I - 2\vec{u}\vec{u}^T$; here, $I$ denotes the $n \times n$ identity matrix and $\vec{u}^T$ denotes the transpose of the vector $\vec{u}$.

(a) Compute $M^T - M$ and $M^2$.

(b) Use part (a) to show that if $\lambda$ is an eigenvalue of $M$, then $\lambda \in \{-1, 1\}$.

(c) Find an eigenvector of $M$ corresponding to the eigenvalue $-1$ and an eigenvector of $M$ corresponding to the eigenvalue 1.

**6.** (a) Straightforward that $M^T = M$ and $M^2 = I$.

(b) For example, if $\lambda$ is such that $M\vec{w} = \lambda\vec{w}$, $\vec{w} \neq \vec{0}$, then $M^2\vec{w} = \lambda^2\vec{w}$ and since $M^2 = I$ we get $\lambda^2 = 1$. Since $M$ is symmetric we know that $\lambda \in \mathbb{R}$ and so $\lambda = \pm 1$. Alternately, one can also appeal to the known result that the eigenvalues of an orthogonal matrix are of modulus 1, but here the instructions were clear as to use (a) to arrive to the conclusion...

(c) $M\vec{u} = -\vec{u}$ and for any non-zero $\vec{v}$ orthogonal to $\vec{u}$, we have $M\vec{v} = \vec{v}$.

**Problem 7.** Let $n \in \mathbb{N}$ and let $V$ be an $n$-dimensional vector space over $\mathbb{C}$. Suppose $T : V \to V$ is a linear transformation which has the property that there exists an integer $m \geq 1$ such that $T^m = 0$.

(1) Show that $T$ has an eigenvector corresponding to the eigenvalue 0.
(2) Prove, using induction, that there exists a basis for $V$ such that the matrix of $T$ with respect to this basis is strictly upper triangular.

(1) Let $k > 0$ be the minimum positive integer such that $T^k = 0$. If $k = 1$ then $T = 0$ and any nonzero vector will do. Otherwise, $T^{k-1} \neq 0$ and there exists a vector $w$ such that $T^{k-1}w \neq 0$. Therefore, $T(T^{k-1}w) = 0$ and we may choose $v = T^{k-1}w$.

(2) We proceed by induction on the dimension of $V$. If the dimension of $V = 0$ then $T$ is scalar multiplication so the hypotheses imply $T = 0$ and the result follows. Now suppose $\dim V = n > 1$ and that the result holds on any vector space $U$ of dimension $< n$. To employ the induction hypothesis, we note that $T|_{\operatorname{im} T}$ maps to $\operatorname{im} T$. By the rank-nullity theorem and the previous part, the dimension of $\operatorname{im} T$ is less than $n$ so that there exists a basis $v_1, \ldots v_m$ of $\operatorname{im} T$ such that the matrix for $T|_{\operatorname{im} T}$ with respect to this basis is strictly upper triangular. If $w_1, \ldots, w_k$ is a basis for $\ker T$, then, as one can check, the matrix for $T$ with respect to $w_1, \ldots, w_k, v_1, \ldots, v_m$ is strictly upper triangular, as desired.

**Problem 8.** Find the general solution of the system of differential equations

$$\begin{cases} \frac{dx}{dt} &= x + 1 \\ \frac{dy}{dt} &= x^2 - y. \end{cases}$$

**8.** Solve first $x' = x + 1$ by integrating $\int dx/(x+1) = \int dt$ to get $x(t) = k_1 e^t - 1, k_1 \in \mathbb{R}$. Now, the second equation is

$$y' = -y + (k_1 e^t - 1)^2 = -y + 1 - 2k_1 e^t + k_1^2 e^{2t}.$$

To solve it, we find the generic solution of the homogenous part $y' = -y$ which is $k_2 e^{-t}, k_2 \in \mathbb{R}$, and then a particular solution of the form $\alpha + \beta e^t + \gamma e^{2t}$. Substituting in the equation in $y$ we get the particular solution $1 - k_1 e^t + \frac{k_1^2}{3} e^{2t}$. All in all, we conclude that the general solution is

$$x(t) = k_1 e^t - 1, y(t) = k_2 e^{-t} + 1 - k_1 e^t + \frac{k_1^2}{3} e^{2t}, k_1, k_2 \in \mathbb{R}.$$

**Problem 9.** The sequence $(a_n)_{n=1}^{\infty}$ is defined recursively by

$$a_1 = 1, \; a_{n+1} = \left(1 + \frac{1}{n}\right)^{-n} a_n.$$

(a) Show that $(a_n)$ is bounded and monotonic, and compute $\lim_{n\to\infty} a_n$.

(b) Find the radius of convergence of the series $\sum_{n=1}^{\infty} a_n x^{2n}$.

**9.** (a) Simply note that all terms $a_n > 0$ and that $a_{n+1}/a_n < 1$. Thus, $(a_n)$ is non-increasing. This gives that $a_n$ is bounded below by 0 and above by 1. To find the limit, let $\ell = \lim a_n$ and pass to the limit in the recursion to get $\ell = e^{-1}\ell$, i.e., $\ell = 0$.

(b) We use the ratio test. We compute

$$\lim_{n\to\infty} \frac{|a_{n+1}x^{2n+2}|}{|a_n x^{2n}|} = e^{-1}x^2.$$

Thus, we need $e^{-1}x^2 < 1$, i.e., $|x| < e^{-1/2}$. The radius of convergence is $e^{-1/2}$.

**Problem 10.** Let $\sum_{n=1}^{\infty} x_n$ be a convergent series of positive numbers. Show that the series $\sum_{n=1}^{\infty} \cos(x_n)$ is divergent and that the series $\sum_{n=1}^{\infty} \sin(x_n)$ is convergent.

**10.** Since the series converges, we know $x_n \to 0$ as $n \to \infty$. In particular, for $n \geq N$ sufficiently large we have $x_n \in (0, \pi/2)$. Now, since $\cos(x_n) \to \cos(0) = 1$ as $n \to \infty$, we see that the series of cosines diverges. Also, since for $n \geq N$ we have $0 < \sin(x_n) < x_n$, a direct comparison of the sine series with the original one gives its convergence.