Qualifying Exam

Spring 2011
March 28, 2011

Number ________________

YOU MAY USE CALCULATORS FOR THIS EXAM. BE ADVISED HOWEVER THAT EVERY QUESTION CAN BE ANSWERED WITHOUT THE USE OF A CALCULATOR AND MORE THAN LIKELY CAN BE ANSWERED MORE EFFICIENTLY WITHOUT THE USE OF A CALCULATOR.

Problem 1. For which value or values of a are the graphs of $y = ax^2$ and $y = \ln x$ tangent? (This means that the graphs have a common point and the same tangent line at that point.)

Problem 2. Let $a \in \mathbb{R}$. Consider the function $g_a(x) = \dfrac{a - x}{1 - (1 - a)x}$.

(a) Show that g_a is its own inverse.

(b) Prove that g_a is decreasing on each open interval on which it is defined.

(c) Let $a > 0$. Prove that g_a maps $[0, a]$ onto itself as a bijection.

Problem 3. Find a 2×2 matrix A with the following three properties: (a) A has real nonzero entries, (b) A is symmetric, (c) the eigenvalues of A are 0 and 1.

Problem 4. In this problem we will define two special functions.

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases} \qquad \text{and} \qquad \operatorname{Si}(x) = \int_0^x \operatorname{sinc}(t)\, dt.$$

(a) Find the Maclaurin series for the function Si. The Maclaurin series is the Taylor series centered at 0.

(b) Calculate $\displaystyle\lim_{x \to 0} \frac{x - \operatorname{Si}(x)}{x^3}$.

Problem 5. Let n be a positive integer and let a, b, c, d be real numbers. Consider the $(2n+1) \times (2n+1)$ matrix given on the right and answer the following questions.

$$\begin{bmatrix} a & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & b \\ 0 & a & \cdots & 0 & 0 & 0 & \cdots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 & b & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & c & 0 & d & \cdots & 0 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c & \cdots & 0 & 0 & 0 & \cdots & d & 0 \\ c & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & d \end{bmatrix}$$

(a) Find a necessary and sufficient condition for the given matrix to be invertible.

(b) Assume that the condition for invertibility is satisfied. Give the formula for the inverse.

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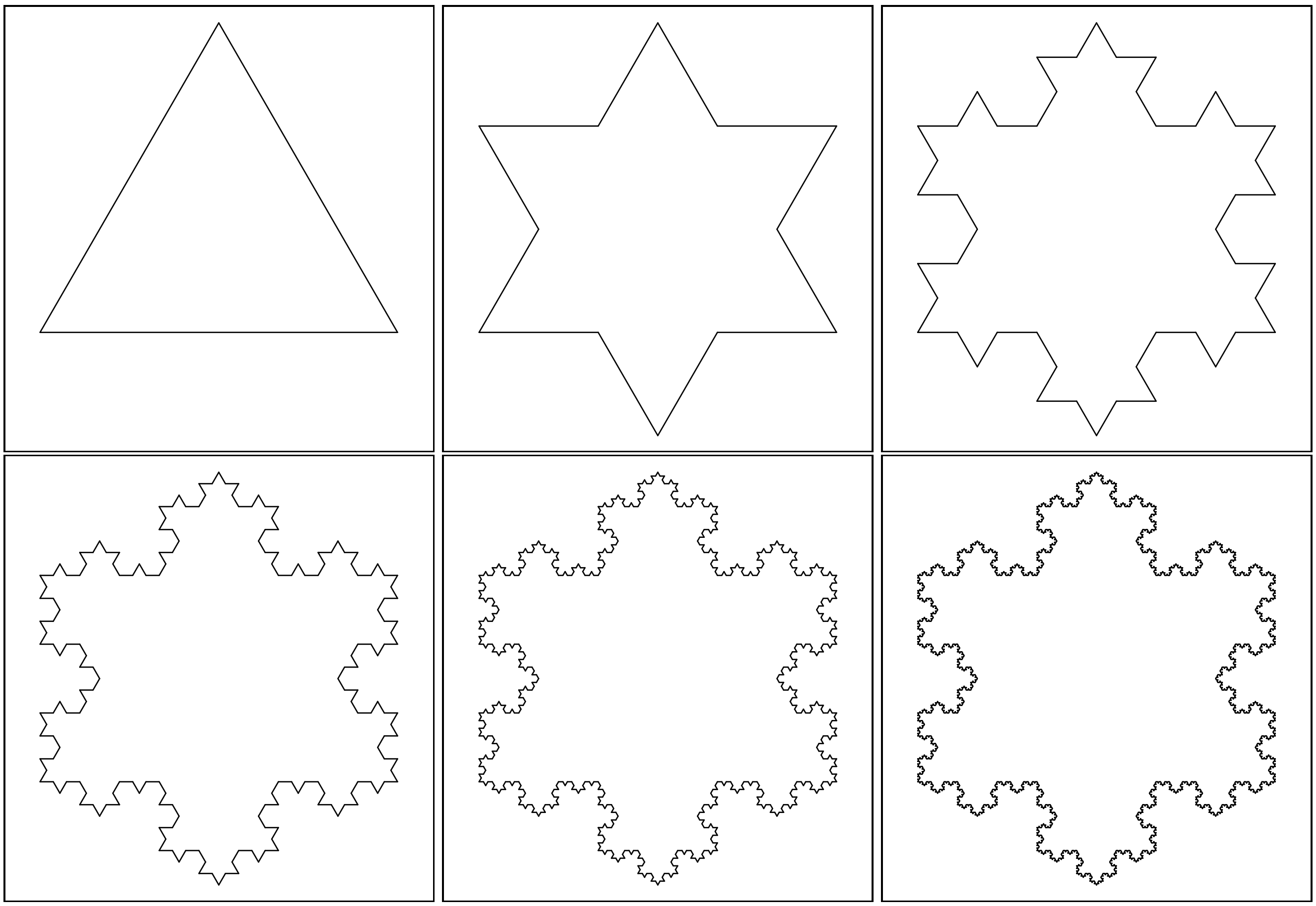
Problem 6. The Koch snowflake is a figure that is constructed in the following way: The Koch snowflake at step 0, denoted by K_0, is an equilateral triangle with sides of length 1. Then, on each step we break every side of the figure into three equal segments, on every middle segment we build an equilateral triangle (facing outwards from our figure) and finally we throw the middle segments away. The Koch snowflakes $K_0, K_1, K_2, K_3, K_4, K_5$ are given in the figures below. Calculate the area A_n enclosed by the Koch snowflake K_n and determine the limit of A_n as $n \to +\infty$.



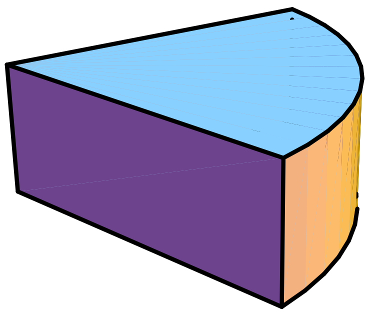
Problem 7. This is a mathematical loan problem. Assume that the interest on this loan is compounded continuously and the payments are made continuously. This assumption allows us to use differential equations to model this loan. The annual interest rate on this loan is 6%. (This is $0.5\% = \frac{1}{200}$ monthly rate.) The payments are made continuously with the monthly rate of $1000\,\dfrac{e^{3/2}}{e^{3/2}-1}$ dollars. (This is approximately \$1,287.22 dollars monthly payment.) The current amount of this loan is \$200,000 dollars.

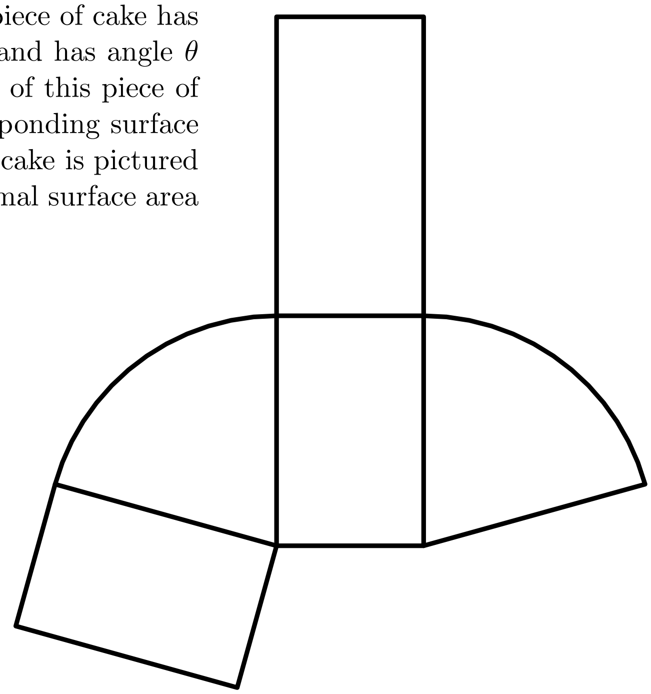
(a) Set up an initial value problem which models this loan.

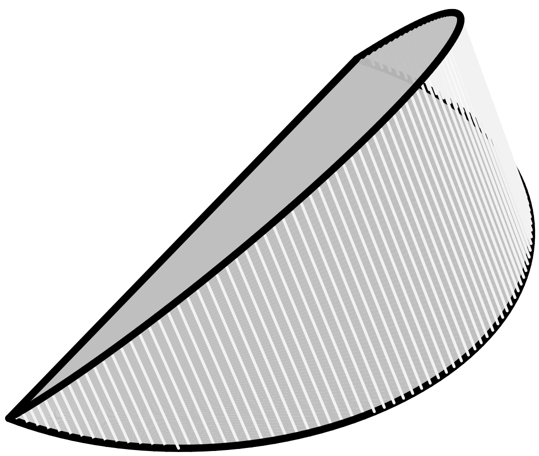
(b) Solve the stated initial value problem.

(c) How long will it take for this loan to be paid off? Please answer in years rather than months.

Problem 8. A piece of cake is shown below. This piece of cake has height z, is cut from a cylindrical cake of radius r and has angle θ at the center of the piece. Assume that the volume of this piece of cake is 1. Calculate z, r and θ for which the corresponding surface area of the cake is minimal. The surface area of the cake is pictured to the right. **Note:** You can assume that such minimal surface area exists. You do not need to prove that.







Problem 9. The picture on the left shows a simplified taco. It is a flat unit disk tortilla which is folded along its diameter so that the flat semi-disks planes form an angle of $\pi/4$. Determine the volume of the filling that can be fitted in this taco if the filling has to be lined up with the edges of the semi-disks, as shown in the figure to the left.

Problem 10. A unit disk is divided in nine parts by two pairs of parallel lines. The lines are at the same distance from the center and the pairs of parallel lines are mutually orthogonal. See the figure on the right. Is it possible to choose the lines in such a way that all nine parts have identical areas?