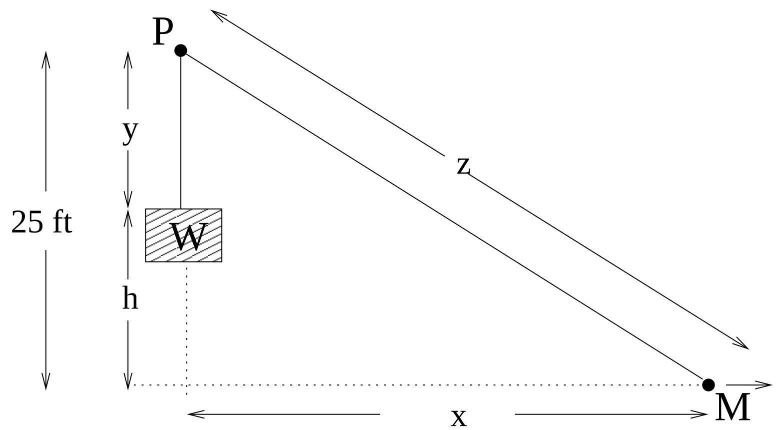
Qualifying Exam Solutions

September 2009

You may use calculators for this exam. Show all work to receive full credit.

1. The following figure shows a rope running through a pulley at P, bearing a weight W at one end. The other end of the rope is at point M, where it is being pulled along the ground, away from the pulley, at a rate of 6 ft/sec. Suppose that P is 25 feet above the ground and that the rope is 45 feet long. How fast is the weight being raised at the instant when the distance x is 15 feet?



Solution: First, we find the values of z and y when $x = 15$:

$x^2 + 25^2 = z^2$, so, when $x = 15$, we have $z = 5\sqrt{34} = 29.15475$. Then, since $z + y = 45$, we have $y = 45 - z = 45 - 5\sqrt{34} = 15.845$.

Next, we find the value of dy/dx when $x = 15$: Note that $y + z = 45$ and that $x^2 + 25^2 = z^2$. Then $x^2 + 25^2 = (45 - y)^2$. By implicit differentiation, we find $\frac{dy}{dx} = \frac{-x}{45-y}$. When $x = 15$, this gives $\frac{dy}{dx} = \frac{-3}{\sqrt{34}} = -0.514495$.

Finally, when $x = 15$, we have $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = (-0.514495)(6) = -3.08697$ft/sec.

OR

Note that $dy/dt = -1$ since $y = 45 - z$. Also $dz/dx = x/\sqrt{x^2 + 25^2}$ since $z = \sqrt{x^2 + 25^2}$

$$\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dx} \cdot \frac{dx}{dt} = -1 \cdot \frac{x}{\sqrt{x^2 + 25^2}} \cdot 6 = -\frac{6x}{\sqrt{x^2 + 25^2}} \overset{a}{=} -\frac{6 \cdot 15}{\sqrt{850}} = -\frac{18}{\sqrt{34}}.$$

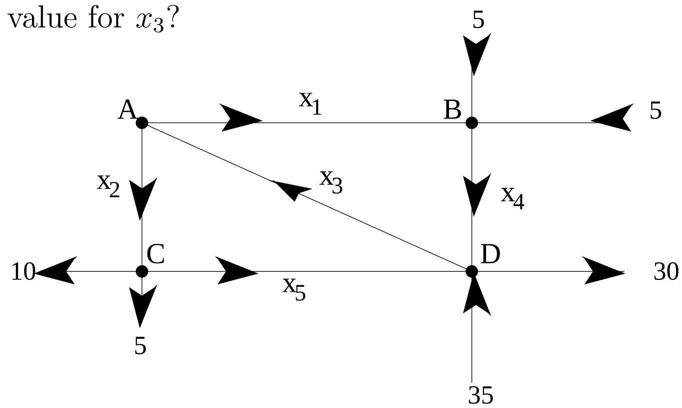
[a]when $x = 15$.

OR

$$z^2 \;=\; 25^2 + x^2 \implies 2z\frac{dz}{dt} = 2x\frac{dx}{dt} \implies \frac{dz}{dt} = \frac{18}{\sqrt{34}},$$

then note that $dy/dt = -dz/dt$.

2. Consider the following figure showing traffic patterns with all measurements given in cars per minute. Assuming that none of the flows can be negative, what is the smallest possible value for x_3?



Solution: At each node/intersection, the total rate flowing into the intersection must equal the total rate flowing out. This gives us the equations (node A) $x_1 + x_2 = x_3$, (node B) $x_1 + 10 = x_4$, (node C) $15 + x_5 = x_2$, and (node D) $35 + x_4 + x_5 = 30 + x_3$. These in turn give us the following augemented matrix:

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & -10 \\ 0 & 1 & 0 & 0 & -1 & 15 \\ 0 & 0 & 1 & -1 & -1 & 5 \end{bmatrix} \overset{rref}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 10 \\ 0 & 1 & 0 & 0 & -1 & 15 \\ 0 & 0 & 1 & -1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In parametric vector form, the solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_4 - 10 \\ x_5 + 15 \\ x_4 + x_5 + 5 \\ \text{free} \\ \text{free} \end{bmatrix}$$

Note that, in order for x_1 to be nonnegative, x_4 must be at least 10. There is no similar direct constraint on x_5. Then, since $x_3 = 5 + x_4 + x_5$, it must be that $x_3 \geq 15$.

3. Find the mass of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ if the density at any point is proportional to the distance from the point to the z-axis and the density at point (1,1/2, 5/4) is $3\sqrt{5}/4$.

   Solution: First we find the constand of proportionality. Let $d(x, y)$ be the density at point (x, y). Then

$$d(x,y) = k\sqrt{x^2+y^2} \implies \frac{3\sqrt{5}}{4} = k\sqrt{\frac{5}{4}} \implies k = \frac{3}{2}.$$

   The density is given by $f(x, y, z) = \alpha\sqrt{x^2 + y^2}$, where $\alpha = 3.2$ is a constant of proportionality. Thus we have

$$M = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=x^2+y^2}^{4} \alpha\sqrt{x^2+y^2}dzdydx.$$

   A change to cylindrical coordinates (noting that $\alpha\sqrt{x^2 + y^2} = \alpha r$) gives

$$\begin{aligned} M &= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2}^{4}(\alpha r)rdzdrd\theta \\ &= \ldots = \frac{128}{15}\pi\alpha = \frac{64\pi}{5}. \end{aligned}$$

4. Find the maximum value of $f(x, y, z) = xyz$ among all points (x, y, z) lying on the line of intersection of planes $x + y + z = 30$ and $x + y - z = 0$.

   Solution: Note that the intersection between planes $x+y+z = 30$ and $x+y-z = 0$ consists of points such that $z = 15$ and $x+y = 15$ (I got this by adding the equations for the two planes). So that reduces the problem to: Maximize $f(x, y) = 15xy$ subject to $x+y = 15$. That's easy: substitute $y = 15-x$, and now the problem is: Maximize $f(x) = 15x(15 - x)$. It maximizes at $x = 15/2$. Since $y = 15 - x$, we now have $y = 15/2$. $z = 15$ was found earlier. The maximum value is $f(x, y, z) = xyz = (15/2)(15/2)(15) = 15^3/4 = 3375/4 = 843.75$.

5. Each year 1/10 of the people outside Metropolis move to Metropolis and 2/10 of the people inside Metropolis move out. At time 0 there are 10 million people living in Metropolis and 200 million living outside.

   (a) Determine a 2×2 matrix A so that the vectors $\begin{bmatrix} m_k \\ o_k \end{bmatrix}$ of people living inside and outside Metropolis in year k are related by $\begin{bmatrix} m_{k+1} \\ o_{k+1} \end{bmatrix} = A \begin{bmatrix} m_k \\ o_k \end{bmatrix}$.

   Solution:

$$A = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}$$

   (b) Express the population vector in year 0 as a linear combination of the eigenvectors of A.

   Solution: The eigenvalues are found as follows:

$$\begin{vmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{vmatrix} = (0.9-\lambda)(0.8-\lambda) - 0.02 = \lambda^2 - 1.7\lambda + 0.7.$$

   Applying the quadratic equation, we find that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0.7$.

   The eigenvectors, $\vec{e}_1$ and $\vec{e}_2$, are found as follows:

$$\begin{aligned} 0 &= A\vec{e}_1 - \lambda_1 I\vec{e}_1 = \begin{bmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{bmatrix} \vec{e}_1 \implies \vec{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ 0 &= A\vec{e}_2 - \lambda_2 I\vec{e}_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} \vec{e}_2 \implies \vec{e}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

   We want to write (10, 200) as a linear combination of $\vec{e}_1$ and $\vec{e}_2$. The equation to be solved is:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 10 \\ 200 \end{bmatrix}.$$

   The solution is

$$\vec{x}_0 = \begin{bmatrix} 10 \\ 200 \end{bmatrix} = 70\vec{e}_1 - 60\vec{e}_2.$$

   Answers will, of course, vary a bit depending on which eigenvectors were chosen.

(c) Determine the number of people living in Metropolis in year k as a function of k. What will happen to the population of Metropolis eventually?

Solution: The key thing to note here is that $A^t e_i = \lambda_i^t e_i$. The number of people in and out of Metropolis at time t is

$$\begin{aligned} A^t\vec{x}_0 &= A^t(70\vec{e}_1 - 60\vec{e}_2) = 70A^t\vec{e}_1 - 60A^t\vec{e}_2 \\ &= 70\lambda_1^t\vec{e}_1 - 60\lambda_2^t\vec{e}_2 = 70\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 60(0.7)^t\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 70-60(0.7)^t \\ 140+60(0.7)^t \end{bmatrix} \end{aligned}$$

The number of people living outside of Metropolis at time t is given by $140 + 60(0.7)^t$. The number living inside is $70 - 60(0.7)^t$.

$$\lim_{t\to\infty}\begin{bmatrix} m_t \\ o_t \end{bmatrix} = \begin{bmatrix} 70 \\ 140 \end{bmatrix}$$

6. Let $p_0, p_1, p_2, \ldots$ be a sequence of real numbers. For any function $g : \mathbb{R} \to \mathbb{R}$, define a function T such that $T(g(x)) = \sum_{x=0}^{\infty} g(x)p_x$. Suppose $p_x = \frac{\lambda^x}{x!}$, where λ is a positive constant. Derive closed-form expressions for $T(x)$ and $T(x^2)$. **Solution:**

$$\begin{aligned} T(x) &= \sum_{x=0}^{\infty} xp_x = \sum_{x=0}^{\infty} x\frac{\lambda^x}{x!} = \sum_{x=1}^{\infty} x\frac{\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} \\ &\overset{(j=x-1)}{=} \sum_{j=0}^{\infty} \frac{\lambda^{(j+1)}}{j!} = \lambda\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \lambda e^{\lambda} \\ T(x^2) &= \sum_{x=0}^{\infty} x^2 p_x = \sum_{x=0}^{\infty} x^2\frac{\lambda^x}{x!} = \sum_{x=1}^{\infty} x^2\frac{\lambda^x}{x!} = \sum_{x=1}^{\infty} x\frac{\lambda^x}{(x-1)!} \\ &\overset{(j=x-1)}{=} \sum_{j=0}^{\infty} \frac{(j+1)\lambda^{(j+1)}}{j!} = \left(\sum_{j=0}^{\infty}\frac{j\lambda^{(j+1)}}{j!}\right) + \left(\sum_{j=0}^{\infty}\frac{\lambda^{(j+1)}}{j!}\right) \\ &= \lambda\left(\sum_{j=0}^{\infty}\frac{j\lambda^j}{j!}\right) + \lambda\left(\sum_{j=0}^{\infty}\frac{\lambda^j}{j!}\right) \\ &= \lambda T(x) + \lambda e^{\lambda} = \lambda^2 e^{\lambda} + \lambda e^{\lambda} \end{aligned}$$

7. Torricelli's Law states that the rate of flow of fluid out of a container through a hole in the bottom is proportional to the square root of the depth of fluid above the hole. Suppose that a container has the shape of a cone (point downward) with radius 5 feet and height 10 feet, that it is full of water initially, and that the initial rate of flow is 1 ft^3/sec. (It may help to remember that the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.)

   (a) Determine an expression for the depth of water in the container at time t.
   Solution: T's law states that $dV/dt = -k\sqrt{h(t)}$. We know that $h(0) = 10$ and $dV/dt\big|_{t=0} = -1ft^3/sec$, so $k = 1/\sqrt{10}$. Then, T's law becomes $dV/dt = -\sqrt{\frac{h(t)}{10}}$. Note that $r(t) = h(t)/2$ at all times (geometry), so, using the volume equation, we have $V(t) = \frac{\pi}{3}r(t)^2h(t) = \frac{\pi}{3}\frac{h(t)^3}{4} = \frac{\pi}{12}h(t)^3$. From this equation, we have

$$\frac{dV}{dt} = \frac{3\pi}{12}h(t)^2\frac{dh(t)}{dt} = \frac{\pi h(t)^2}{4}\cdot\frac{dh(t)}{dt}$$

   Setting our two expressions for dV/dt equal, we get

$$\begin{aligned} -\sqrt{\frac{h(t)}{10}} &= \frac{\pi h(t)^2}{4}\cdot\frac{dh(t)}{dt} \implies -\frac{4}{\pi\sqrt{10}}dt = h^{(3/2)}dh \\ -\frac{4}{\pi\sqrt{10}}t &= \frac{2}{5}h^{(5/2)} + Constant \\ h^{5/2} &= -\frac{\sqrt{10}}{\pi}t + constant \\ h &= \left(C - \frac{\sqrt{10}}{\pi}t\right)^{2/5} \end{aligned}$$

   When $t = 0$, $h = 10$, so $C = 10^{5/2}$. Our final answer is $h(t) = \left(10^{5/2} - \frac{\sqrt{10}}{\pi}t\right)^{2/5}$.

   (b) For what values of t does the expression represent the depth of water? Explain what the expression says about the depth of water over time and why you chose the range of values of t that you did.
   Solution: At some point, h becomes zero. The equation is only valid up until that point. We set $h(t) = 0$ and solve for t:

$$\begin{aligned} \left(10^{5/2} - \frac{\sqrt{10}}{\pi}t\right)^{2/5} &= 0 \\ t &= \frac{10^{5/2}\pi}{10^{1/2}} = 100\pi. \end{aligned}$$

   The equation is valid for $0 \le t \le 100\pi$.

8. Consider a curve in the first quadrant of $\mathbb{R}^2$ with an interesting property: If you pick any point, $P1$, on the curve, and let $P2$ be the x-intercept of the line tangent to the curve at $P1$, then the y-axis will bisect the line segment from $P1$ to $P2$. The curve contains the point (1,2). Derive the equation of the curve.

   Solution: Let $P1$ have coordinates (x_0, y_0). Define P_b to be the point of bisection on the y-axis. Note that the x-and y-coordinates of P_b must be midway between the x- and y-coordinates of $P1$ and $P2$. Hence, $P2 = (-x_0, 0)$ and $P_b = (0, y_0/2)$.

   We could also find P_b's y-coordinate by noting that it is the y-intercept of the tangent line. The slope of that line is $f'(x_0)$, so its equation must be $y - y_0 = f'(x_0)(x - x_0)$. Moving this into $y = mx + b$ form, we find $b = (y_0 - f'(x_0)x_0)$.

   Setting our two expressions for the y-intercept equal, we have $y_0/2 = y_0 - f'(x_0)x_0$, so $f'(x_0)x_0 = y_0/2$. This is a differential equation. Changing notation, this is:

$$\frac{dy}{dx}x = \frac{y}{2}.$$

   I move things around a little:

$$\begin{aligned} \frac{dy}{y} &= \frac{dx}{2x} \\ \ln(y) &= \frac{\ln(x)}{2} + C_0 \\ y &= Ce^{\ln(x^{1/2})} = Cx^{1/2}. \end{aligned}$$

   We know this curve contains the point (1,2), so we must have $C = 2$. Hence: $y = 2\sqrt{x}$.

9. Let S be the surface $xy^2z + x^2yz^2 = 30$.

   (a) Find the equation of the plane tangent to S at (3,2,1).

   Solution: Let $f(x,y,z) = xy^2z + x^2yz^2 - 30$. Then $\nabla f = (y^2z + 2xyz^2, 2xyz + x^2z^2, xy^2 + 2x^2yz)$. Evaluated at (3,2,1), this becomes $\nabla f = (16, 21, 48)$.

   Let our plane be $ax+by+cz = d$ or, equivalently $g(x,y,z) = ax+by+cz-d = 0$. Since $\nabla g\big|_{(3,2,1)} = \nabla f\big|_{(3,2,1)}$, a, b, and c are already determined ($a = 16$, $b = 21$, $c = 48$). Since $f(3,2,1) = g(3,2,1)$, we have $0 = f(3,2,1) = 3(16) + 2(21) + 1(48) - d$, so $d = 138$. Our plane is then: $16x + 21y + 48z - 138 = 0$.

   (b) Use your answer to (a) to determine in what direction from (3,2,1) you should go in order to reach the surface $xy^2z + x^2yz^2 = 29$ in the shortest distance. Estimate this distance. Explain your answers briefly, using a diagram.

   Solution: The direction to be traveled is that opposite the gradient (because we want to *decrease* $G(x,y,z) = xy^2z + x^2yz^2$), namely, in the direction (-16, -21, -48). Since $|nablaG|$ represents the slope of the curve in the direction of the gradient, a unit move in this direction corresponds roughly to a change of $|\nabla G|$ in G. Since we want to change G by just 1 unit (from 30 to 29) we should go $1/|\nabla G|$ units.

$$\frac{1}{|\nabla G|} = \frac{1}{\sqrt{16^2 + 21^2 + 48^2}} = \frac{1}{54.7184} = 0.01825.$$

10. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $x^2 + 4y^2 + 9z^2 = 9$.

    Solution: We want to maximize $Volume = f(x,y,z) = 8xyz$ subject to $h(x,y,z) = x^2 + 4y^2 + 9z^2 - 9 = 0$. Define λ to be a Lagrange multiplier. Then, at the point (x_0, y_0, z_0) where the maximum occurs, we have $\nabla f(x,y,z) = \lambda \nabla h(x,y,z)$. Note

$$\begin{aligned} \nabla f(x,y,z) &= yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} && (1) \\ \nabla h(x,y,z) &= 2x\mathbf{i} + 8y\mathbf{j} + 18z\mathbf{k} && (2) \end{aligned}$$

    Comparing Equations 1 and 2, we see that

$$\begin{aligned} yz &= 2x\lambda && (3) \\ xz &= 8y\lambda && (4) \\ xy &= 18z\lambda && (5) \end{aligned}$$

    From Equations 3, 4 and 5, we see that $xyz = 2x^2 = 8y^2 = 18z^2$, which yields: $x^2 = 4y^2$ and $x^2 = 9z^2$. Then, since $x^2 + 4y^2 + 9z^2 = 9$, we have $3x^2 = 9$, so $x^2 = 3$ (if we are considering (x,y,z) to be the point in the first octant where our parellelepiped meets the ellipsoid). Then, since $x^2 = 4y^2$, this gives $y = \sqrt{3}/2$, and, similarly, we get $z = \frac{\sqrt{3}}{3}$. The volume is then $A = 8xyz = 4\sqrt{3}$.