Qualifying Exam

Fall 2023
September 18, 2023

Number ____________________

You may use calculators for this exam. — Justify all your answers. Answer specific questions by giving the exact values, not approximations.

Problem 1. Use partial fractions to find the Taylor series about $x = 0$ of the function $\dfrac{1}{x^2 + x - 2}$ and determine its open interval of convergence.

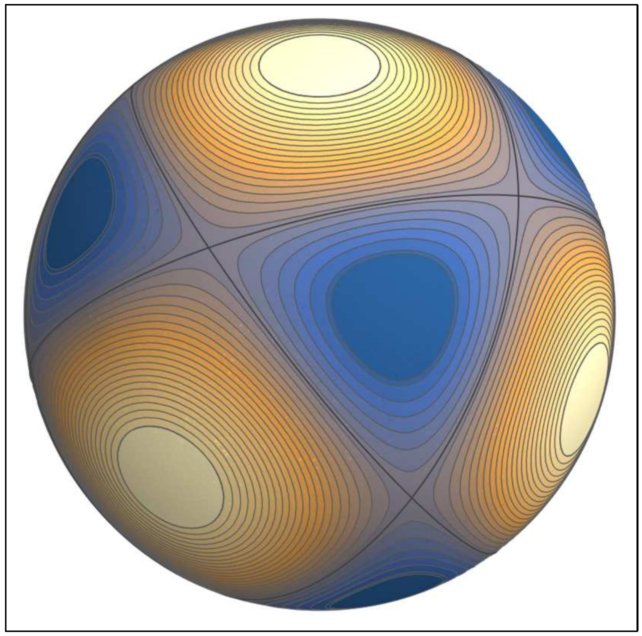
Problem 2. Consider the temperature in $\mathbb{R}^3$ given by

$$T(x, y, z) = x^4 + y^4 + z^4, \quad (x, y, z) \in \mathbb{R}^3.$$

(a) Determine the minimum and the maximum value of the function T on the unit sphere in $\mathbb{R}^3$ centered at the origin.

(b) Find the total number of critical points of the function T on the unit sphere. Determine how many critical points are minimums, maximums and saddle points.

(c) The picture on the right shows a heat map and isotherms (lines of the same temperature) of T on the unit sphere. Using the colors and the isotherms in the picture, describe which visible critical points are minimums, maximums, and saddle points.



Problem 3. Let n be an integer greater than 1 and let A be an $n \times n$ matrix. Let $\mathbf{1} \in \mathbb{R}^n$ be the vector whose all entries equal to 1. Consider the following block matrix

$$M = \begin{bmatrix} A & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}.$$

Here M is an $(n+1) \times (n+1)$ matrix. Recall that an $n \times n$ matrix A is said to be positive definite if for all nonzero vectors $\mathbf{v} \in \mathbb{R}^n$ we have $\mathbf{v}^\top A \mathbf{v} > 0$.

(a) Find a nonsingular 3×3 matrix A such that the corresponding matrix M is singular.

(b) Prove the following implication: If a matrix A is positive definite, then the corresponding matrix M is nonsingular.

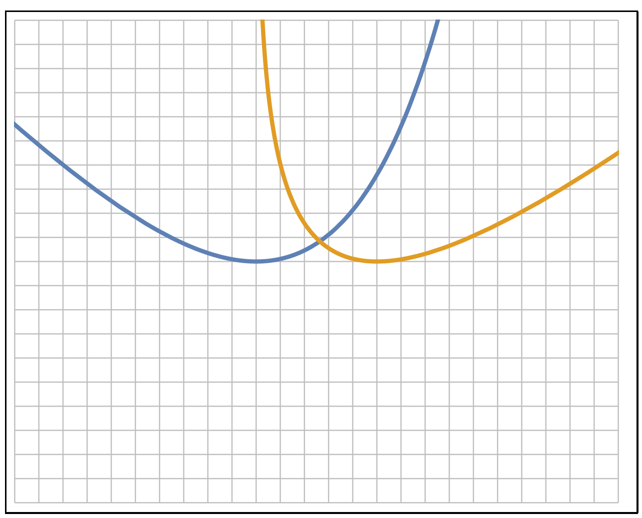
Problem 4. By $\mathbb{R}_+$ we denote the set of positive real numbers. Let $a \in \mathbb{R}_+$ and consider the two functions

$$f_a(x) = -ax + e^x \quad \text{and} \quad g_a(x) = ax - \log(x).$$

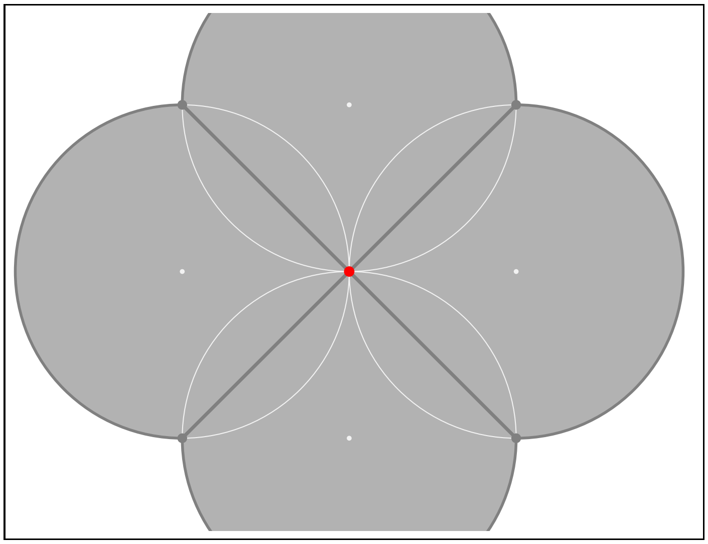
(a) Prove that for every $a \in \mathbb{R}_+$ there exists $s \in \mathbb{R}$ such that $f_a(s) \le f_a(x)$ for all $x \in \mathbb{R}$.

(b) Prove that for every $a \in \mathbb{R}_+$ there exists $t \in \mathbb{R}_+$ such that $g_a(t) \le g_a(x)$ for all $x \in \mathbb{R}_+$.

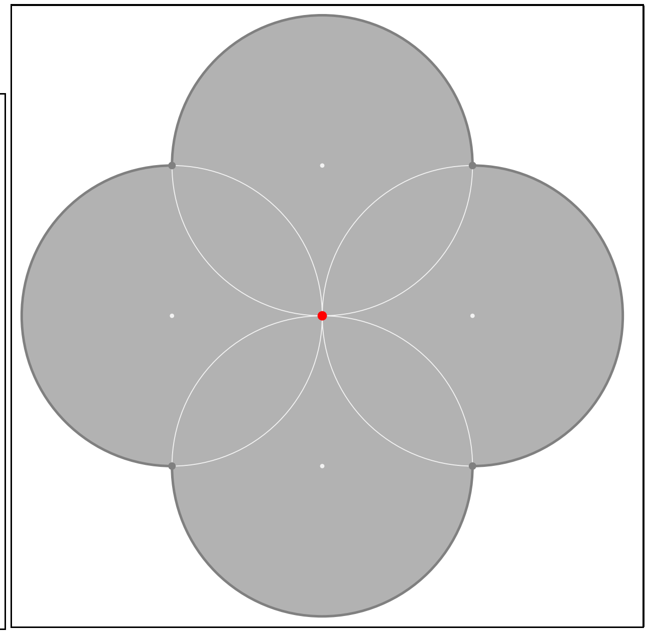
(c) Find the value of $a \in \mathbb{R}_+$ for which the corresponding minimum values of f_a and g_a are equal. That is find the value of $a \in \mathbb{R}_+$ for which the graphs of f_a and g_a look like in the picture.



Problem 5. Consider four unit disks centered at the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$ and the red point positioned at $(0, 0)$, as shown in the picture to the right. In this picture, the union of the four disks is colored gray. Calculate the exact value of the average distance of the red point to a point in the gray area.



Hint: Join the darker gray points and the red point with line segments, and consider the partition of the gray area obtained in this way. See the picture to the left.



Problem 6. Let n be a positive integer and let H_n be the n-th harmonic number, that is

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Prove that for all real numbers p such that $p > 1$ the series

$$\sum_{n=1}^{\infty} \frac{H_n}{n^p}$$

converges.

Problem 7. Consider the sequence of functions

$$x \mapsto \frac{\sin(nx)}{n \sin(x)}, \quad x \in (-\pi, 0) \cup (0, \pi), \quad n \in \mathbb{N}.$$

(a) Prove that for every $n \in \mathbb{N}$ there exist a continuous function $f_n : (-\pi, \pi) \to \mathbb{R}$ such that

$$f_n(x) = \frac{\sin(nx)}{n \sin(x)} \quad \text{for all} \quad x \in (-\pi, 0) \cup (0, \pi).$$

(b) Prove that there exists a function $g : (-\pi, \pi) \to \mathbb{R}$ such that for all $x \in (-\pi, \pi)$ we have

$$\lim_{n \to \infty} f_n(x) = g(x).$$

Plot an accurate graph of the function g and state its range in set notation.

Problem 8. Suppose you are climbing a hill whose shape in xyz-space is given by the equation

$$z = f(x,y) = 1000 - \frac{1}{200}x^2 - \frac{1}{400}y^2$$

where x, y, and z (height) are measured in meters. You are at the point $P = (120, -160, 864)$ on the hill. In the contour plot below, the projection $(120, -160)$ of P onto xy-plane is the blue point.

(a) If you start walking Northeast, will you ascend or descend? With what slope?

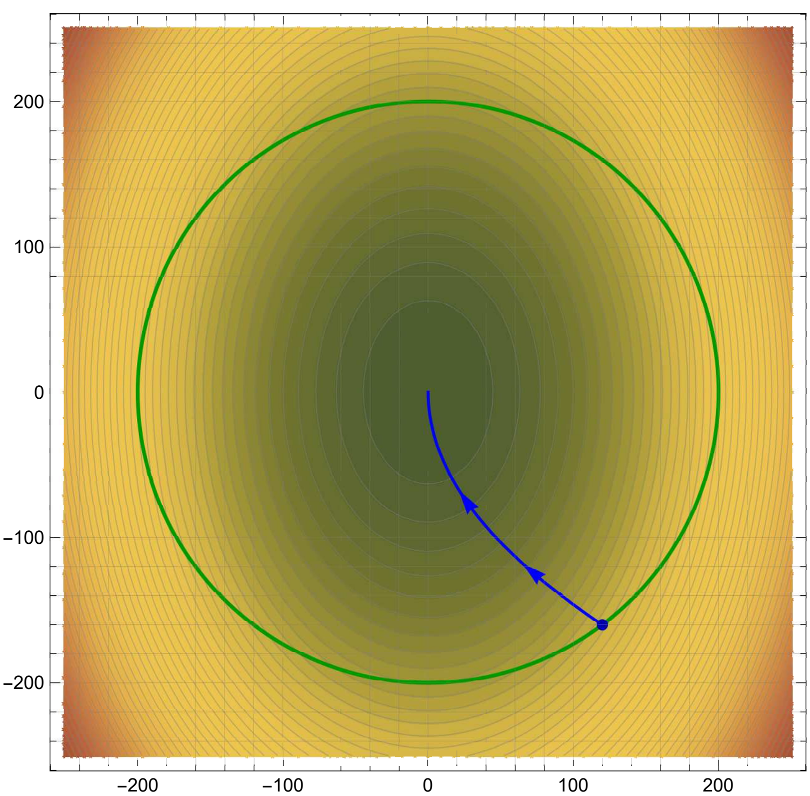
(b) In which direction from the point P is the slope of ascent the largest? What is the rate of ascent in that direction? Express the direction in two different ways:

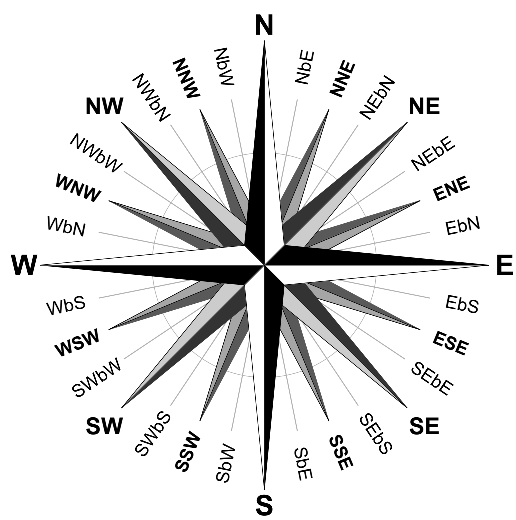
  (i) as a two-dimensional vector in xy-plane,

  (ii) as an approximate direction on the 32-wind compass rose; see the picture of the rose below.

(c) Notice the projection of P onto xy-plane is at the distance 200 meters from the origin $(0,0)$. Consider all the points $\big(x, y, f(x,y)\big)$ on the hill such that $\sqrt{x^2+y^2} = 200$. The projections of those points form the green circle in the contour plot below. Find the maximum rate of ascent for the points described in this paragraph. At what point(s) does this maximum rate of ascent occur?

(d) Recall that the *path of steepest ascent* is a path on the hill that follows the direction of the largest slope of ascent at every point along the path. Show that the projection on the xy-plane of the path of steepest ascent through the point $(120, -160, 864)$ is a part of the parabola $y^2 = ax$, where a is a real number. Determine the exact value of a.





The positive x-axis represents East and the positive y-axis represents North. The picture on the left gives a topographic map of the hill with level curves. Above is the 32-wind compass rose. The abbreviation "NEbN" stands for "Northeast by North". These abbreviations are used in navigation.

Problem 9. Find a symmetric real 3×3 matrix A whose rank is 1, and $A\begin{bmatrix}3 & 2 & 1\end{bmatrix}^\top = \begin{bmatrix}1 & -2 & 2\end{bmatrix}^\top$.

Problem 10. Find the general solution of the system $\frac{dx}{dt} = -2x + 4y^2$, $\frac{dy}{dt} = 3y$.

THE END OF THE EXAM