Qualifying Exam

Fall 2023
September 18, 2023 Number ____________________

YOU MAY USE CALCULATORS FOR THIS EXAM. — JUSTIFY ALL YOUR ANSWERS. ANSWER SPECIFIC QUESTIONS BY GIVING THE EXACT VALUES, NOT APPROXIMATIONS.

Problem 1. Use partial fractions to find the Taylor series about $x = 0$ of the function $\dfrac{1}{x^2+x-2}$ and determine its open interval of convergence.

Solution.

$$\begin{aligned}
\frac{1}{x^2+x-2} &= \frac{1}{3}\left(\frac{1}{x-1} - \frac{1}{x+2}\right) \\
&= \frac{1}{3}\left(\frac{1}{2}\left(\frac{-1}{1-(-x/2)}\right) - \frac{1}{1-x}\right) \\
&= \frac{1}{3}\left(\frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{x}{2}\right)^n - \sum_{n=0}^{\infty} x^n\right) \\
&= \frac{1}{3}\sum_{n=0}^{\infty}\left(\left(-\frac{1}{2}\right)^{n+1} x^n \quad - \quad x^n\right) \\
&= \sum_{n=0}^{\infty}\frac{1}{3}\left(\left(-\frac{1}{2}\right)^{n+1} - \quad 1\right) x^n
\end{aligned}$$

with convergence occurring for the set $\{x : |x/2| < 1\} \cap \{x : |x| < 1\}$. Thus the interval of convergence is $(-1, 1)$.

$\square$

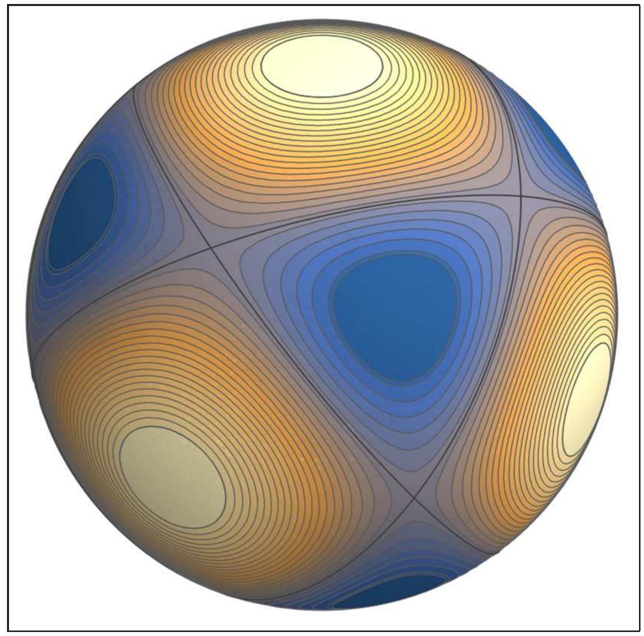
Problem 2. Consider the temperature in $\mathbb{R}^3$ given by

$$T(x, y, z) = x^4 + y^4 + z^4, \quad (x, y, z) \in \mathbb{R}^3.$$

(a) Determine the minimum and the maximum value of the function T on the unit sphere in $\mathbb{R}^3$ centered at the origin.

(b) Find the total number of critical points of the function T on the unit sphere. Determine how many critical points are minimums, maximums and saddle points.

(c) The picture on the right shows a heat map and isotherms (lines of the same temperature) of T on the unit sphere. Using the colors and the isotherms in the picture, describe which visible critical points are minimums, maximums, and saddle points.



Solution. Let $g(x, y, z) = x^2 + y^2 + z^2$, ot that the constrain is $g(x, y, z) = 1$. To apply the Lagrange multiplier method we calculate

$$(\nabla T)(x, y, z) = \langle 4x^3, 4y^3, 4z^3 \rangle, \qquad (\nabla f)(x, y, z) = \langle 2x, 2y, 2z \rangle.$$

To find the critical points of the function T on the unit sphere we solve the system of equations

$$\langle 4x^3, 4y^3, 4z^3 \rangle = \lambda \langle 2x, 2y, 2z \rangle, \quad x^2 + y^2 + z^2 = 1,$$

for $(x, y, z) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R} \setminus \{0\}$. The solutions for λ are $\frac{2}{3}$, 1 and 2 with the corresponding points (x, y, z) as follows:

- for $\lambda = \frac{2}{3}$ we have eight points $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, and the temperature at these points is $\frac{1}{3}$,
- for $\lambda = 1$ we have twelve points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$, $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and the temperature at these points is $\frac{1}{2}$,
- for $\lambda = 2$ we have six points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, $(0, 0, -1)$ and the temperature at these points is 1.

(a) Since the extreme values of the temperature on the unit sphere must occur at the critical points obtained above, we deduce that the maximum temperature is 1, the minimum temperature is $\frac{1}{3}$, and the value $\frac{1}{2}$ is taken at the saddle points.
(b) In the above work, we calculated all the critical points on the unit sphere. There are 26 critical points; 8 minimums, 12 saddle points, and 6 maximums.
(c) As we complete the image of the unit sphere in our imagination, we see that there are six points at which the temperature is encoded with the light yellow color. Therefore the temperature is 1 at those points. There are eight points at which the temperature is encoded with the blue color. Therefore, the temperature is $\frac{1}{3}$ at those points. Notice that in the given picture there are four great circles that is isotherms. These great circles intersect in twelve points which are the saddle points for the temperature. In the neighbourhood of these points the temperature takes values higher than $\frac{1}{2}$ (indicated with a yellowish) and the temperature lower that $\frac{1}{2}$ indicated with the blueish color. □

Problem 3. Let n be an integer greater than 1 and let A be an $n \times n$ matrix. Let $\mathbf{1} \in \mathbb{R}^n$ be the vector whose all entries equal to 1. Consider the following block matrix

$$M = \begin{bmatrix} A & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}.$$

Here M is an $(n+1) \times (n+1)$ matrix. Recall that an $n \times n$ matrix A is said to be positive definite if for all nonzero vectors $\mathbf{v} \in \mathbb{R}^n$ we have $\mathbf{v}^\top A\mathbf{v} > 0$.

(a) Find a nonsingular 3×3 matrix A such that the corresponding matrix M is singular.

(b) Prove the following implication: If a matrix A is positive definite, then the corresponding matrix M is nonsingular.

Solution 1. (a) One possible answer is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

(b) A is positive definite $\Rightarrow$ A is nonsingular. Thus there exists a unique $\mathbf{c} \neq \mathbf{0}$ such that $A\mathbf{c} = \mathbf{1}$.

Suppose M is singular. Then the columns of M are linearly dependent, i.e., for the linear combination, where $\mathbf{A}_i = i$th column of A,

$$d_1 \begin{bmatrix} \mathbf{A}_1 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} \mathbf{A}_2 \\ 1 \end{bmatrix} + \ldots + d_n \begin{bmatrix} \mathbf{A}_n \\ 1 \end{bmatrix} + d_{n+1} \begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix} = \mathbf{0},$$

not all d_i's are zeroes. Note that $d_{n+1} \neq 0$. If this is not so, then $d_1\mathbf{A}_1 + d_2\mathbf{A}_2 + \ldots d_n\mathbf{A}_n = \mathbf{0}$, which implies that $d_1 = d_2 = \ldots = d_n = d_{n+1} = 0$ since A is nonsingular, contradicting the assumption that M is singular.

Thus,

$$e_1 \begin{bmatrix} \mathbf{A}_1 \\ 1 \end{bmatrix} + e_2 \begin{bmatrix} \mathbf{A}_2 \\ 1 \end{bmatrix} + \ldots + e_n \begin{bmatrix} \mathbf{A}_n \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix}$$

where $e_i = -\frac{d_i}{d_{n+1}}$, i.e.,

$$\begin{bmatrix} A \\ \mathbf{1}^\top \end{bmatrix} \mathbf{e} = \begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix}.$$

Now $\mathbf{e}$ must be equal to $\mathbf{c}$, since $A\mathbf{c} = \mathbf{1}$ and $\mathbf{c}$ is unique. It follows that

$$\mathbf{1}^\top \mathbf{c} = \mathbf{c}^\top \mathbf{1} = 0$$

$$\Rightarrow \quad \mathbf{c}^\top A\mathbf{c} = 0.$$

But $\mathbf{c} \neq \mathbf{0}$, contradicting the premise that A is positive definite. □

Solution 2. Let n be a positive integer and let A be a nonsingular $n \times n$ matrix. Then for all $\mathbf{x} \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$ we have

$$\begin{bmatrix} A & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x} = \mathbf{0},$$

and

$$\begin{bmatrix} A & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \alpha = 0.$$

Therefore,

$$\begin{bmatrix} A & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} \neq \mathbf{0} \;\text{ and }\; \alpha \neq 0.$$

Consequently,

$$\begin{bmatrix} A & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x} = -\alpha A^{-1}\mathbf{1} \;\text{ and }\; \mathbf{1}^\top A^{-1}\mathbf{1} = 0. \tag{1}$$

The equivalence in (1) yields the following equivalence: The matrix $\begin{bmatrix} A & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}$ is singular if and only if $\mathbf{1}^\top A^{-1}\mathbf{1} = 0$. That is, $\begin{bmatrix} A & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}$ is singular if and only if the nonzero vectors $\mathbf{1}$ and $A^{-1}\mathbf{1}$ are orthogonal.

We first answer (b). Assume that A is positive definite. Then A^{-1} is also positive definite. Therefore $\mathbf{1}^\top A^{-1}\mathbf{1} > 0$. Thus, the vectors $\mathbf{1}$ and $A^{-1}\mathbf{1}$ are not orthogonal. By the equivalence in the last sentence of the preceding paragraph, we deduce that the matrix $\begin{bmatrix} A & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}$ is nonsingular.

To answer (a) we need to find a nonsingular 3×3 matrix A such that $\mathbf{1}^\top A^{-1}\mathbf{1} = 0$. Since it is easiest to calculate the inverse of a diagonal matrix we will find $a, b, c \in \mathbb{R} \setminus \{0\}$ such that we have

$$0 = [111] \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Hence,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is a specific solution for (a). The last claim is confirmed by

$$\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\square$

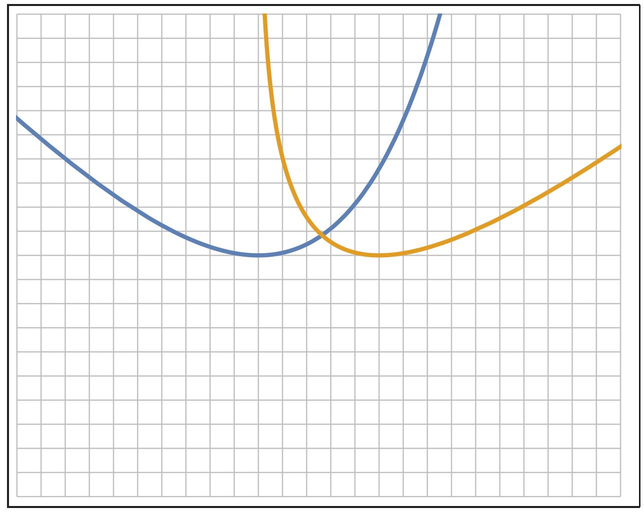
Problem 4. By $\mathbb{R}_+$ we denote the set of positive real numbers. Let $a \in \mathbb{R}_+$ and consider the two functions

$$f_a(x) = -ax + e^x \quad \text{and} \quad g_a(x) = ax - \log(x).$$

(a) Prove that for every $a \in \mathbb{R}_+$ there exists $s \in \mathbb{R}$ such that $f_a(s) \leq f_a(x)$ for all $x \in \mathbb{R}$.

(b) Prove that for every $a \in \mathbb{R}_+$ there exists $t \in \mathbb{R}_+$ such that $g_a(t) \leq g_a(x)$ for all $x \in \mathbb{R}_+$.

(c) Find the value of $a \in \mathbb{R}_+$ for which the corresponding minimum values of f_a and g_a are equal. That is find the value of $a \in \mathbb{R}_+$ for which the graphs of f_a and g_a look like in the picture.



Solution. (a) Let $a \in \mathbb{R}_+$ be fixed. Since $f_a'(x) = -a + e^x$, the function f_a has only one critical point $s = \ln a$. Since the value of the second derivative of f_a at $s = \ln a$ equals $a > 0$, the function f_a takes its minimum at $s = \ln a$. The minimum value is $f_a(s) = a - a\ln a = a(1 - \ln a)$.

(b) Let $a \in \mathbb{R}_+$ be fixed. Since $g_a'(x) = a - 1/x$, the function g_a has only one critical point $t = 1/a$. Since the value of the second derivative of g_a at $t = 1/a$ equals $a^{-2} > 0$, the function g_a takes its minimum at $t = 1/a$. The minimum value is $g_a(t) = 1 + \ln a$.

(c) We need to solve the following equation for a

$$1 + \ln a = a - a\ln a.$$

We rewrite the equation as

$$(1 + a)\ln a = a - 1$$

and further

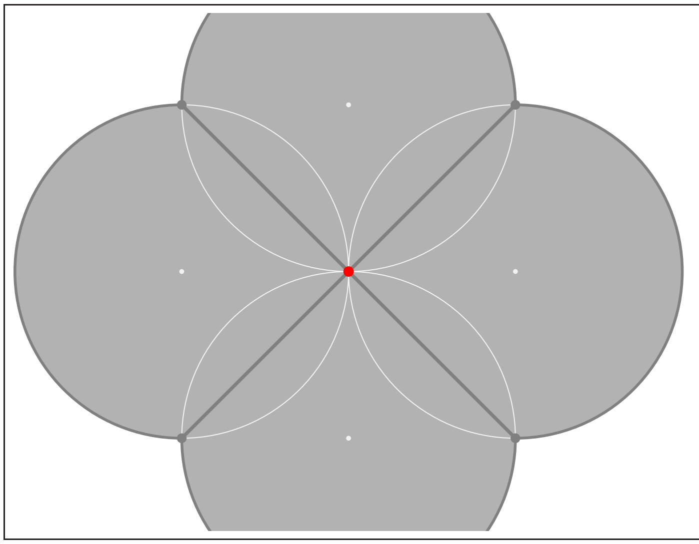
$$\ln a = \frac{a-1}{a+1}$$

It is clear that for both functions $\ln a$ and $\frac{a-1}{a+1}$ have zero for $a = 1$. Since the derivative of the function

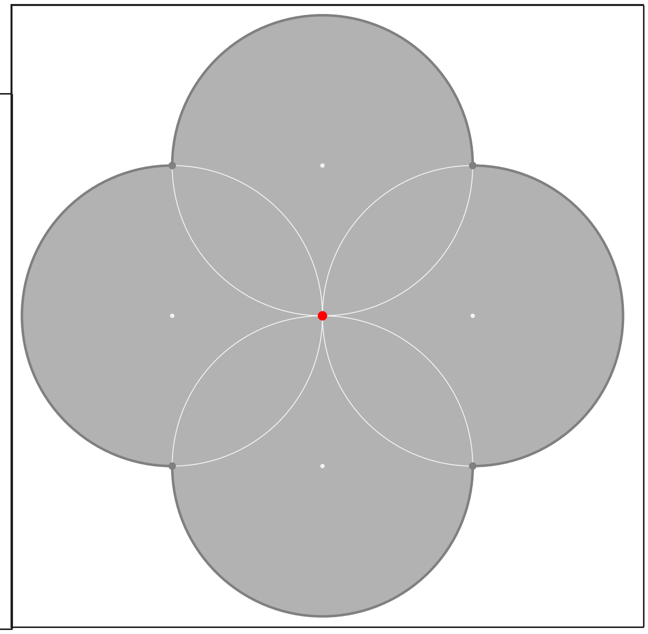
$$a \mapsto \ln a - \frac{a-1}{a+1}, \qquad a > 0,$$

is strictly positive, the value $a = 1$ is the only value for which functions f_a and g_a have the same minimum. □

Problem 5. Consider four unit disks centered at the points $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$ and the red point positioned at $(0,0)$, as shown in the picture to the right. In this picture, the union of the four disks is colored gray. Calculate the exact value of the average distance of the red point to a point in the gray area.



Hint: Join the darker gray points and the red point with line segments, and consider the partition of the gray area obtained in this way. See the picture to the left.



Solution. Denote by Q one the four congruent pieces that the gray area partitions into, following the hint. Q consists of an isosceles right triangle of the area 1 and a half of a unit disk. Thus, the area of Q is $1+\pi/2$. The average distance that the problem seeks is the same as the average distance of a point in Q to the red point. By definition the average distance is

$$\frac{1}{\operatorname{area} Q}\iint_Q \sqrt{x^2+y^2}\,dxdy.$$

We use the polar coordinates and Fubini's theorem to calculate this integral:

$$\begin{aligned}
\iint_Q \sqrt{x^2+y^2}\,dxdy &= \int_{-\pi/4}^{\pi/4}\int_0^{2\cos\theta} r^2\,dr\,d\theta \\
&= \frac{8}{3}\int_{-\pi/4}^{\pi/4} (\cos\theta)^3\,d\theta \\
&= \frac{8}{3}\int_{-\pi/4}^{\pi/4} \big(1-(\sin\theta)^2\big)\big(\cos\theta\big)\,d\theta \\
&= \frac{8}{3}\int_{-\pi/4}^{\pi/4} \big(1-(\sin\theta)^2\big)\big(\cos\theta\big)\,d\theta \\
&= \frac{8}{3}\int_{-\pi/4}^{\pi/4} (\cos\theta)\,d\theta - \frac{8}{3}\int_{-\pi/4}^{\pi/4} (\sin\theta)^2\,d(\sin\theta) \\
&= \frac{8\sqrt{2}}{3} - \frac{4\sqrt{2}}{9} = \frac{20\sqrt{2}}{9}
\end{aligned}$$

Thus, the average value is $\dfrac{\frac{20\sqrt{2}}{9}}{1+\frac{\pi}{2}} = \dfrac{40\sqrt{2}}{9(2+\pi)} \approx 1.22246.$ ☐

Problem 6. Let n be a positive integer and let H_n be the n-th harmonic number, that is

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Prove that for all real numbers p such that $p > 1$ the series

$$\sum_{n=1}^{\infty} \frac{H_n}{n^p}$$

converges.

Solution. Using the same idea as the integral test, note that for every $n \in \mathbb{N}$

$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \quad \leq \quad 1 + \int_1^n \frac{1}{x} dx = 1 + \ln(n).$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{H_n}{n^p} \leq \sum_{1}^{\infty} \frac{1 + \ln(n)}{n^p} \leq 1 + \int_1^{\infty} \frac{1 + \ln(x)}{x^p} dx = 1 + \frac{p}{(1-p)^2}$$

for $p > 1$. Thus, $\sum_{n=1}^{\infty} \frac{H_n}{n^p}$ converges by the integral test.

Another approach is to note that, since $p > 1$, there exists an $\epsilon > 0$ such that $p - \epsilon > 1$. For this ϵ, $\lim_{n\to\infty} \frac{\ln(n)}{n^\epsilon} = 0$ (which can be shown by, say, the l'Hopital's rule), which means that there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$, $\frac{\ln(n)}{n^\epsilon} < 1$ and

$$\frac{\ln(n)}{n^p} = \frac{\ln(n)}{n^\epsilon} \cdot \frac{1}{n^{p-\epsilon}} \leq \frac{1}{n^{p-\epsilon}}.$$

This implies that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p}$ converges by the comparison test and the well-known property of the harmonic series. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} + \sum_{n=1}^{\infty} \frac{\ln(n)}{n^p} = \sum_{1}^{\infty} \frac{1 + \ln(n)}{n^p}$$

also converges, and likewise for $\sum_{n=1}^{\infty} \frac{H_n}{n^p}$.

□

Problem 7. Consider the sequence of functions

$$x \mapsto \frac{\sin(nx)}{n\sin(x)}, \quad x \in (-\pi,0)\cup(0,\pi), \quad n \in \mathbb{N}.$$

(a) Prove that for every $n \in \mathbb{N}$ there exist a continuous function $f_n : (-\pi,\pi) \to \mathbb{R}$ such that

$$f_n(x) = \frac{\sin(nx)}{n\sin(x)} \quad \text{for all} \quad x \in (-\pi,0)\cup(0,\pi).$$

(b) Prove that there exists a function $g : (-\pi,\pi) \to \mathbb{R}$ such that for all $x \in (-\pi,\pi)$ we have

$$\lim_{n\to\infty} f_n(x) = g(x).$$

Plot an accurate graph of the function g and state its range in set notation.

Solution. (a) Let $n \in \mathbb{N}$ be arbitrary. Since the function $x \mapsto \frac{\sin(nx)}{n\sin(x)}$ is continuous on the open set $(-\pi,0)\cup(0,\pi)$, and

$$\lim_{x\to 0} \frac{\sin(nx)}{n\sin(x)} = \lim_{x\to 0} \frac{\frac{\sin(nx)}{nx}}{\frac{\sin(x)}{x}} = \frac{\lim_{x\to 0}\frac{\sin(nx)}{nx}}{\lim_{x\to 0}\frac{\sin(x)}{x}} = 1,$$

the function $f_n : (-\pi,\pi) \to \mathbb{R}$ defined piecewise by

$$f_n(x) = \begin{cases} \frac{\sin(nx)}{n\sin(x)}, & \text{if} \quad x \in (-\pi,0)\cup(0,\pi), \\ 1, & \text{if} \quad x = 0, \end{cases}$$

is continuous on $(-\pi,\pi)$.

(b) Let $x \in (-\pi,\pi)$ and calculate $\lim\limits_{n\to\infty} f_n(x)$. Consider two cases: $x = 0$ and $x \in (-\pi,0)\cup(0,\pi)$. If $x = 0$, then

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 1 = 1.$$

Let $x \in (-\pi,0)\cup(0,\pi)$ be arbitrary. Then $|\sin x| > 0$. Since for all $n \in \mathbb{N}$ we have $|\sin(nx)| \le 1$, we have

$$\forall n \in \mathbb{N} \qquad \left|\frac{\sin(nx)}{n\sin(x)}\right| = \frac{|\sin(nx)|}{n|\sin(x)|} \le \frac{1}{n|\sin x|}. \tag{2}$$

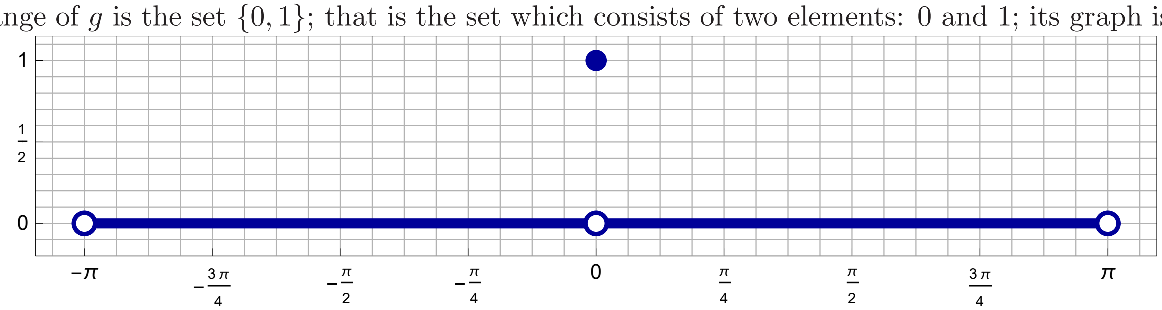
Since $|\sin x| > 0$ is a fixed positive number, (2) and the squeeze theorem yield

$$\lim_{n\to\infty} f_n(x) = 0.$$

Set

$$g(x) = \begin{cases} 0, & \text{if} \quad x \in (-\pi,0)\cup(0,\pi), \\ 1, & \text{if} \quad x = 0. \end{cases}$$

The range of g is the set $\{0,1\}$; that is the set which consists of two elements: 0 and 1; its graph is below.



□

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Problem 8. Suppose you are climbing a hill whose shape in xyz-space is given by the equation

$$z = f(x,y) = 1000 - \frac{1}{200}x^2 - \frac{1}{400}y^2$$

where x, y, and z (height) are measured in meters. You are at the point $P = (120, -160, 864)$ on the hill. In the contour plot below, the projection $(120, -160)$ of P onto xy-plane is the blue point.

(a) If you start walking Northeast, will you ascend or descend? With what slope?

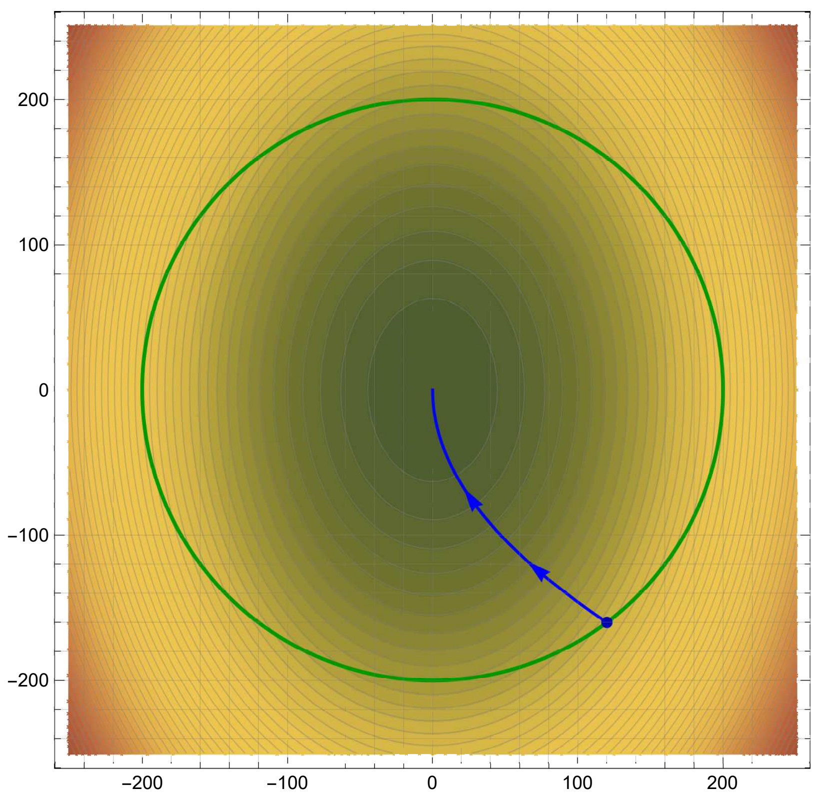
(b) In which direction from the point P is the slope of ascent the largest? What is the rate of ascent in that direction? Express the direction in two different ways:

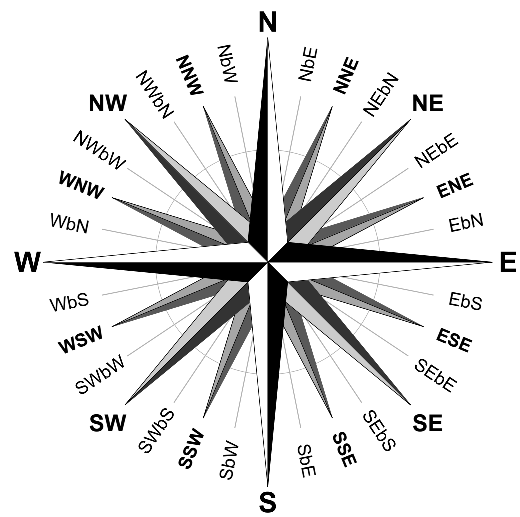
  (i) as a two-dimensional vector in xy-plane,

  (ii) as an approximate direction on the 32-wind compass rose; see the picture of the rose below.

(c) Notice the projection of P onto xy-plane is at the distance 200 meters from the origin $(0,0)$. Consider all the points $\big(x, y, f(x,y)\big)$ on the hill such that $\sqrt{x^2+y^2} = 200$. The projections of those points form the green circle in the contour plot below. Find the maximum rate of ascent for the points described in this paragraph. At what point(s) does this maximum rate of ascent occur?

(d) Recall that the *path of steepest ascent* is a path on the hill that follows the direction of the largest slope of ascent at every point along the path. Show that the projection on the xy-plane of the path of steepest ascent through the point $(120, -160, 864)$ is a part of the parabola $y^2 = ax$, where a is a real number. Determine the exact value of a.





The positive x-axis represents East and the positive y-axis represents North. The picture on the left gives a topographic map of the hill with level curves. Above is the 32-wind compass rose. The abbreviation "NEbN" stands for "Northeast by North". These abbreviations are used in navigation.

Solution. It will be helpful to calculate the gradient of f:

$$(\nabla f)(x,y) = \left\langle -\frac{x}{100}, -\frac{y}{200} \right\rangle.$$

The gradient vector at the point $(120, -160)$ is

$$(\nabla f)(120, -160) = \left\langle -\frac{120}{100}, \frac{160}{200} \right\rangle = \left\langle -\frac{6}{5}, \frac{4}{5} \right\rangle.$$

(a) The unit vector in the direction Northeast is

$$\left\langle \cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right)\right\rangle = \frac{\sqrt{2}}{2}\langle 1,1\rangle.$$

Therefore, if we proceed in the direction Northeast the rate of change of altitude will be

$$\frac{\sqrt{2}}{2}\big((\nabla f)(120,-160)\big)\cdot\langle 1,1\rangle = \frac{\sqrt{2}}{2}\left\langle -\frac{6}{5},\frac{4}{5}\right\rangle\cdot\langle 1,1\rangle = -\frac{\sqrt{2}}{5} \approx -0.282843.$$

Thus, if one proceeds in the direction Northeast one would be descending with the slope of $-\sqrt{2}/5$.

(b) The gradient gives the direction of steepest ascent and the norm of the gradient vector gives the slope of the steepest ascent. As we calculated, the gradient is

$$\left\langle -\frac{6}{5},\frac{4}{5}\right\rangle, \qquad \left\|\left\langle -\frac{6}{5},\frac{4}{5}\right\rangle\right\| = \frac{2\sqrt{13}}{5} \approx 1.44222.$$

The unit vector in the direction of the gradient is

$$\left\langle -\frac{3}{\sqrt{13}},\frac{2}{\sqrt{13}}\right\rangle = \langle \cos\alpha, \sin\alpha\rangle, \quad \text{where} \quad \alpha = \arccos\left(-\frac{3}{\sqrt{13}}\right) \approx 2.55359.$$

To determine which is the closest direction on the 32-wind compass rose we need to calculate the approximate radian measures of the nearby compass directions: Nothwest is $3\pi/4 \approx 2.35619$, Northwest-by-West is $13\pi/16 \approx 2.55254$ and West-Northwest is $7\pi/8 \approx 2.74889$. Thus, the closest direction on the 32-wind compass rose is Northwest-by-West.

(c) The points on the circle are given by

$$(200\cos\theta, 200\sin\theta), \qquad \theta \in [0,2\pi).$$

To find the maximum rate of ascent at these points, we need to calculate the norm of the gradients at all those points:

$$\left\|\left\langle -\frac{200\cos\theta}{100}, -\frac{200\sin\theta}{200}\right\rangle\right\| = \|\langle -2\cos\theta, -\sin\theta\rangle\| = \sqrt{4(\cos\theta)^2+(\sin\theta)^2} = \sqrt{3(\cos\theta)^2+1}.$$

Clearly the largest value of the preceding expression is $\sqrt{3+1}=2$, i.e., the maximum rate of ascent is 2, which occurs at $\theta=0$ and $\theta=\pi$; i.e., the points at which the ascent is steepest are $(200,0)$ and $(-200,0)$.

(d) To determine the path of the steepest ascent from the point $(120,-160)$, we need to solve the linear system of differential equations:

$$x' = -\frac{x}{100}, \quad y' = -\frac{y}{200}, \quad x(0)=120, \quad y(0)=-160.$$

The solution is

$$x(s) = 120e^{-s/100}, \quad y(s) = -160e^{-s/200}, \quad s \geq 0.$$

Clearly

$$y^2 = 160^2 e^{-s/100} = \frac{160^2}{120}120e^{-s/100} = \frac{640}{3}x.$$

$\square$

Problem 9. Find a symmetric real 3×3 matrix A whose rank is 1, and $A \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^\top = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}^\top$.

Solution 1. Since the rank of A is 1 the column space of A is one-dimensional. Since

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \in \mathrm{Col}(A),$$

we deduce that each column of A is a scalar multiple of $\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}^\top$. Therefore, there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$A = \begin{bmatrix} \alpha & \beta & \gamma \\ -2\alpha & -2\beta & -2\gamma \\ 2\alpha & 2\beta & 2\gamma \end{bmatrix}.$$

Since A is symmetric we must have

$$\beta = -2\alpha, \qquad \gamma = 2\alpha.$$

Thus,

$$A = \alpha \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}.$$

Now the condition $A \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^\top = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}^\top$ yields $\alpha = 1$. □

Solution 2. Since A is symmetric it is orthogonally diagonalizable. Since the rank of A is 1, A has one nonzero eigenvalue of multiplicity 1. The eigenspace corresponding to the nonzero eigenvalue is the column space of A. Therefore $\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}^\top$ is an eigenvector of A corresponding to the nonzero eigenvalue. The orthogonal complement of $\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}^\top$ is the eigenspace corresponding to the eigenvalue 0. Since

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}^\perp = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\},$$

the orthogonal diagonalization of A is

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Now we use the condition $A \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^\top = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}^\top$ to calculate λ:

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \frac{\lambda}{3} \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

Thus $\lambda = 9$. So,

$$A = 9 \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}.$$

□

Problem 10. Find the general solution of the system $\frac{dx}{dt} = -2x + 4y^2,\ \frac{dy}{dt} = 3y$.

Solution. Since the second equation does not contain the unknown function x, we solve it first. The general solution of $\frac{dy}{dt} = 3y$ is

$$y(t) = Be^{3t}.$$

where B is an arbitrary constant. Now the first equation becomes

$$x'(t) = -2x(t) + 4B^2e^{6t}.$$

This is a first-order nonhoogeneous equation which we solve by using an integrating factor e^{2t}:

$$x'(t)e^{2t} + 2x(t)e^{2t} = 4B^2e^{8t}.$$

The last equation can be rewritten as

$$\frac{d}{dt}\left(x(t)e^{2t}\right) = 4B^2e^{8t},$$

which is solved by indefinite integration

$$x(t)e^{2t} = \frac{1}{2}B^2e^{8t} + A,$$

where A is an arbitrary constant. Thus, the general solution of the given system is

$$x(t) = Ae^{-2t} + \frac{1}{2}B^2e^{6t}, \qquad y(t) = Be^{3t}.$$ □

THE END OF THE EXAM