Number: ______________________

Graduate Qualifying Exam – Spring 2023

Directions: In all problems you must show your work in order to receive credit. You may not use a calculator. All electronic devices must be turned off. You have three hours.

Problem 1. Let $f_a(x, y) = 4x^2 + axy + 3y^2 - 17$ and consider the region

$$V_a := \{(x, y, z) : z \leq f_a(x, y)\}.$$

(a) Determine a vector (in terms of a) that is normal to V_a at $(2, 1, 2a + 2)$. The vector you find should be pointing out of V_a.

(b) Determine all values of a such that the normal vector at $(2, 1, 2a+2)$ you found forms an obtuse angle with the vector $(a, -a, 1)^T$.

Solution. The gradient of f_a is given as $(8x + ay, 6y + ax)^T$. We're looking for an upward normal to the surface $z - f_a(x, y) = 0$. At the given point, normal vectors are scalar multiples of $(-16 - a, -6 - 2a, 1)^T$.

An obtuse angle will be formed if the dot product of $(-16 - a, -6 - 2a, 1)$ with $(a, -a, 1)^T$ is negative. So we need a to satisfy $0 > -16a - a^2 + 6a + 2a^2 + 1 = a^2 - 10a + 1$. The dot product is zero when $a = \frac{10\pm\sqrt{96}}{2}$. Between these values, $a^2 - 10a + 1$ is negative by checking any number between the two roots.

Problem 2. Let V denote the vector space of 2×2 matrices over $\mathbb{R}$. Fix $A \in V$ and define $T : V \to V$ to be the function $T(B) = AB - BA$.

(a) Prove that T is a linear transformation.

(b) Prove that $\operatorname{rank} T \leq 2$.

(c) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Find the eigenvalues of T, and a basis of V consisting of eigenvectors of T.

Solution. To prove that T is linear, let $c_1, c_2 \in \mathbb{R}$ and let $B_1, B_2 \in V$ and consider

$$\begin{aligned} T(c_1B_1 + c_2B_2) &= A(c_1B_1 + c_2B_2) - (c_1B_1 + c_2B_2)A \\ &= c_1AB_1 + c_2AB_2 - c_1B_1A - c_2B_2A \\ &= c_1(AB_1 - B_1A) + c_2(AB_2 - B_2A) \\ &= c_1T(B_1) + c_2T(B_2). \end{aligned}$$

Next, to show $\operatorname{rank} T \leq 2$, it suffices, since $\dim V = 4$, to prove that $\operatorname{nullity} T \geq 2$. To this end, since A commutes with itself and I, we have $I, A \in \operatorname{Nul} T$. Either $\{I, A\}$ is dependent or independent. If it is dependent, then $A = cI$ for some $c \in \mathbb{R}$ so every matrix in V commutes with A so that $\operatorname{Nul} T$ has dimension four. Otherwise, $\dim \operatorname{Nul} T \geq 2$ as desired.

Finally, we compute the matrix of T with respect to the basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. It is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the eigenvalues of T are $0, 1, -1$ and a basis of eigenvectors is the standard basis of V given above.

Problem 3. An aspen's height in meters at time t is given by the function $h(t)$. The height function satisfies $h''(t) = 22.5 - 5h'(t) - 2.25h(t)$ and a particular tree is estimated as growing at an annual rate of 0.5 meters when it is 1 meters tall. Find this tree's height as a function of t.
Solution. The characteristic polynomial of the homogeneous equation is $r^2 + 5r + 2.25$ which has roots $-4.5, -0.5$. Thus, the homogeneous equation has general solution $h_1(t) = C_1e^{-4.5t} + C_2e^{-0.5t}$. We guess a particular solution for the nonhomogeneous equation as being a constant function $h_2(t) = A$; the only such A must be 10 in order to satisfy the nonhomogeneous equation. Our general solution is $h(t) = 10+C_1e^{-4.5t}+C_2e^{-0.5t}$. If we set $h(0) = 1$ and $h'(0) = 0.5$, we get that $-9 = C_1+C_2$ and $0.5 = -4.5C_1 - 0.5C_2$. This means $C_1 = 1, C_2 = -10$, giving $h(t) = 10 + e^{-4.5t} - 10e^{-0.5t}$.

Problem 4. Let V be region in $\mathbb{R}^3$ enclosed by the surface $z = x + y$ and the planes $\{x = 0\}, \{x = 2\}, \{y = 0\}, \{y = 2\}$. The air temperature at any point in $\mathbb{R}^3$ is given by $T(x, y, z) = x + zxy$.
(a) Determine the volume of V.
(b) Find the average air temperature in V.
Solution. We compute $|V| = \int_0^2 \int_0^2 \int_0^2 \int_0^{x+y} dxdydz = \int_0^2 \int_0^2 (x + y)dxdy = 8$.
The average temperature is given by $\frac{1}{|V|} \int_0^2 \int_0^2 \int_0^2 [x + zxy]dxdydz$. Iterating through this gives

$$\begin{aligned} \frac{1}{|V|}\int_0^2\int_0^2\int_0^{x+2y}[x+zxy]dxdydz &= \tfrac{1}{8}\int_0^2\int_0^2[x(x+y)+\tfrac{xy(x+y)^2}{2}]dxdy \\ &= \tfrac{1}{8}\int_0^2[x^3+\tfrac{14}{3}x^2+4x]dx \\ &= \tfrac{220}{72} \end{aligned}$$

Problem 5. Let V be a finite-dimensional vector space over $\mathbb{R}$ and let $L : V \to V$ denote a linear transformation. let $x \in V$ be a nonzero vector and let

$$W = \operatorname{Span}\{x, L(x), L^2(x), \ldots\}.$$

(a) Prove that there is a minimum $j \in \mathbb{Z}$ such that $\{x, L(x), \ldots, L^j(x)\}$ is dependent, and deduce from this that $L^j(x) \in \operatorname{Span}\{x, L(x), \ldots, L^{j-1}(x)\}$.
(b) Prove that $\{x, L(x), \ldots, L^{j-1}(x)\}$ is a basis for W.
(c) Suppose $V = \mathbb{R}^3$ and let A denote the matrix of L with respect to the standard basis for $\mathbb{R}^3$. Suppose that $\operatorname{rank} A = 1$ and x is not in either the column space of A or the null space of A. Compute $\dim W$.

Solution. Since V is finite-dimensional of dimension d, V contains no independent subsets of size $> d$. In particular, $\{x, L(x), \ldots, L^d(x)\}$ is linearly dependent, so there is a minimum j such that $\{x, L(x), \ldots, L^j(x)\}$ is dependent. Thus, since $\{x, L(x), \ldots, L^{j-1}(x)\}$ is independent, there is a dependence relation

$$b_0x + b_1L(x) + \cdots + b_{j-1}L^{j-1}(x) + b_jL^j(x) = 0$$

with $b_j \neq 0$. In particular, we may write $L^j(x)$ as a linear combination of $x, L(x), \ldots, L^{j-1}(x)$.
For the second part, we note that the set of vectors is independent by definition of j, so we need only show it spans W. To do this, it suffices to show that for each $m \in \mathbb{N}$, $L^{j-1+m}(x)$ is in

Span$\{x, L(x), \ldots, L^{j-1}(x)\}$. We prove this by induction on m. By the first part of the problem, the result holds when $m = 1$. For the general case,

$$\begin{aligned} L^{j-1+m+1}(x) &= L(L^{j-1+m}(x)) \\ &= L(\sum_{i=0}^{j-1} c_i L^i(x)) \\ &= \sum_{i=0}^{j-1} c_i L^{i+1}(x) \end{aligned}$$

so that the result follows from the first part of the problem.

For the last part, since x is not in the column space or the null space of A, $\{x, L(x)\}$ is linearly independent and $\{L(x)\}$ is a basis for the column space of A since the rank $A = 1$. it follows that since $L^j(x)$ is in the column space of A for all $j \geq 1$, $\dim W = 2$.

Problem 6. Two vertices of a trapezoid are at $(-2, 0)$ and $(2, 0)$, and the other two lie on the semicircle $x^2 + y^2 = 4$, $y \geq 0$. What is the maximum possible area of the trapezoid? (Recall that the area of a trapezoid with bases b_1 and b_2 and height h is $h(b_1 + b_2)/2$).

Solution. We write the area, $A(x)$ of the trapezoid as a function of x: The location of the two base vertices allows us to set $b_1 = 4$, and since the other base must be parallel and is constrained to the given semicircle, the coordinates of the other vertices are of the form $(-x, h)$ and (x, h). By the given constraint, $h = \sqrt{4 - x^2}$. Thus, the area is

$$A(x) = \frac{1}{2}\sqrt{4 - x^2}(2x + 4).$$

We optimize this function on $[0, 2]$, but note that neither endpoint will give a valid configuration. We find $A'(x) = \frac{-x(x+2)}{\sqrt{4-x^2}} + \sqrt{4 - x^2}$. Setting this equal to zero yields $-x(x + 2) + 4 - x^2 = 0$ which implies that $x^2 + x - 2 = 0$. Thus, the only critical point in $(0, 2)$ is $x = 1$. We find that $A(1) = 3\sqrt{3}$.

Problem 7. The Lanczos derivative of a function $f(x)$ at a point a is defined as

$$f_L'(a) := \lim_{h \mapsto 0^+} \frac{3}{2h^3} \int_{-h}^{h} t f(a + t) dt$$

provided the limit exists.

(a) Let $a \in \mathbb{R}$. Compute $f_L'(a)$ when $f(x) = x$ and when $f(x) = x^2$.
(b) Compute $f_L'(0)$ when $f(x) = |x|$. How does this compare to $f'(0)$?
(c) Does the mean value theorem hold for $f(x) = |x|$ on the interval $[-1, 3]$ if the usual derivative is replaced by the Lanczos derivative?

Solution. First, let $f(x) = x$. Then

$$\begin{aligned} f_L'(a) &= \lim_{h \mapsto 0^+} \frac{3}{2h^3} \int_{-h}^{h} t(a + t) dt \\ &= \lim_{h \mapsto 0^+} \frac{3}{2h^3} \left(\frac{at^2}{2} + \frac{t^3}{3} \right) \Big|_{-h}^{h} \\ &= \lim_{h \mapsto 0^+} \frac{3}{2h^3} \frac{2h^3}{3} \\ &= 1 \end{aligned}$$

Similarly, one shows that if $f(x) = x^2$, $f_L'(a) = 2a$.

Next, suppose $f(x) = |x|$. Then

$$\begin{aligned} f_L'(0) &= \lim_{h\mapsto 0^+} \frac{3}{2h^3}\int_{-h}^{h} t|t|dt \\ &= \lim_{h\mapsto 0^+} \frac{3}{2h^3}\left(\int_0^h t^2dt + \int_{-h}^{0} -t^2dt\right) \\ &= \lim_{h\mapsto 0^+} \frac{3}{2h^3}\left(\frac{t^3}{3}\Big|_0^h + -\frac{t^3}{3}\Big|_{-h}^0\right) \\ &= 0 \end{aligned}$$

In contrast, the ordinary derivative of $f(x)$ at 0 does not exist.

The mean value theorem does not hold since $\frac{f(3)-f(-1)}{4} = \frac{1}{2}$, while, by parts (a) and (b), $f_L'(c)$ can only equal either $1, -1$ or 0.

Problem 8. Do the following series converge or diverge? Completely justify your answer.

(a)

$$\sum_{n=1}^{\infty}(\sqrt{n+1} - \sqrt{n}).$$

(b)

$$\sum_{n=1}^{\infty}(\sqrt[n]{n} - 1)^n.$$

Solution. We show the first series diverges. For, since

$$\begin{aligned} \sqrt{n+1} - \sqrt{n} &= (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}} \end{aligned}$$

and $\frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{2\sqrt{n+1}}$ for $n \geq 1$, then by the comparison test, it suffices to show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. This follows from the p-series test with $p = 1/2$.

Now we show that the second series converges. Since, for $n \geq 2$, $\sqrt[n]{n} > 1$, it suffices, by the root test, to show

$$\lim_{n\to\infty}(\sqrt[n]{n} - 1) < 1.$$

For this, it suffices to show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. To this end, we note that

$$\ln(\lim_{n\to\infty} \sqrt[n]{n}) = \lim_{n\to\infty} \ln(\sqrt[n]{n}) = \lim_{n\to\infty} \frac{\ln n}{n}$$

by continuity of $\ln(x)$. The last expression is equal to 0 by L'Hopital. Thus, since $\ln(\lim_{n\to\infty} \sqrt[n]{n}) = 0$, $\lim_{n\to\infty} \sqrt[n]{n} = 1$ as desired.

Problem 9. Let $F(x, y) = \int_0^x e^{-5t-3y}dt - \int_{x/2}^{y} e^{-3t}dt$.

(a) Find a linear approximation of F at $(0, 1/3)$.

(b) Identify the points on the unit circle where this linear approximation is maximized and where it is minimized. It is not necessary to obtain a simplified expression for these points and you do not have to specify which is the maximizer or minimizer.

Solution. The fundamental theorem of calculus lets us compute the gradient of F as $\nabla F = \begin{pmatrix} e^{-5x-3y} + e^{-3x/2}/2 \\ \frac{-3e^{-3y}}{5}[1-e^{-5x}] - e^{-3y} \end{pmatrix}$. Thus, the linear approximation is $F(0,1/3)+\nabla F(0,1/3)^T(x,y-1/3)^T = \frac{1}{3}[1-e^{-1}] + (e^{-1}+1/2, -e^{-1}) \cdot (x, y-1/3)$. To find the maximizer or minimizer, the Lagrange multiplier rule states that we just need a point on $1-x^2-y^2 = 0$ where $\nabla F(0,1/3)$ is parallel to the gradient of $1-x^2-y^2$. Since the latter is just $(-2x,-2y)^T$, we need unit vectors that are multiples of $\nabla F(0,1/3)$. That means the points are $\pm\frac{(e^{-1}+1/2,-e^{-1})}{\sqrt{(e^{-1}+1/2)^2+e^{-2}}}$.

Problem 10. Let $f(x) = x^{\frac{-x}{x-1}}$ be defined on $(0,1)$.
(a) Use $\ln z \leq z-1$ to show $f(x) \geq e^{-1}$ on $(0,1)$.
(b) Compute $\lim_{x\to 1^-} f(x)$.

Solution. We can rewrite f as $e^{\frac{-x}{x-1}\ln(x)}$. The inequality implies that $\ln 1/x \leq \frac{1}{x} - 1 = \frac{1-x}{x}$. Then $-\ln x \leq \frac{1-x}{x}$, which gives the lower bound we want. For the second part, we can use the inequality again to get that $\ln x \leq x-1$; since $x < 1$, we get that $\ln(x)/(x-1) \geq 1$, so $f(x) \leq e^{-x}$. By the squeeze theorem, the limit is e^{-1}.