Number: ______________________

Graduate Qualifying Exam – Spring 2023

Directions: In all problems you must show your work in order to receive credit. You may not use a calculator. All electronic devices must be turned off. You have three hours.

Problem 1. Let $f_a(x, y) = 4x^2 + axy + 3y^2 - 17$ and consider the region

$$V_a := \{(x, y, z) : z \le f_a(x, y)\}.$$

(a) Determine a vector (in terms of a) that is normal to V_a at $(2, 1, 2a + 2)$. The vector you find should be pointing out of V_a.

(b) Determine all values of a such that the normal vector at $(2, 1, 2a+2)$ you found forms an obtuse angle with the vector $(a, -a, 1)^T$.

Problem 2. Let V denote the vector space of 2×2 matrices over $\mathbb{R}$. Fix $A \in V$ and define $T : V \to V$ to be the function $T(B) = AB - BA$.

(a) Prove that T is a linear transformation.

(b) Prove that $\operatorname{rank} T \le 2$.

(c) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Find the eigenvalues of T, and a basis of V consisting of eigenvectors of T.

Problem 3. An aspen's height in meters at time t is given by the function $h(t)$. The height function satisfies $h''(t) = 22.5 - 5h'(t) - 2.25h(t)$ and a particular tree is estimated as growing at an annual rate of 0.5 meters when it is 1 meters tall. Find this tree's height as a function of t.

Problem 4. Let V be region in $\mathbb{R}^3$ enclosed by the surface $z = x + y$ and the planes $\{x = 0\}, \{x = 2\}, \{z = 0\}, \{y = 0\}, \{y = 2\}$. The air temperature at any point in $\mathbb{R}^3$ is given by $T(x, y, z) = x + zxy$.

(a) Determine the volume of V.

(b) Find the average air temperature in V.

Problem 5. Let V be a finite-dimensional vector space over $\mathbb{R}$ and let $L : V \to V$ denote a linear transformation. let $x \in V$ be a nonzero vector and let

$$W = \operatorname{Span}\{x, L(x), L^2(x), \ldots\}.$$

(a) Prove that there is a minimum $j \in \mathbb{Z}$ such that $\{x, L(x), \ldots, L^j(x)\}$ is dependent, and deduce from this that $L^j(x) \in \operatorname{Span}\{x, L(x), \ldots, L^{j-1}(x)\}$.

(b) Prove that $\{x, L(x), \ldots, L^{j-1}(x)\}$ is a basis for W.

(c) Suppose $V = \mathbb{R}^3$ and let A denote the matrix of L with respect to the standard basis for $\mathbb{R}^3$. Suppose that $\operatorname{rank} A = 1$ and x is not in either the column space of A or the null space of A. Compute $\dim W$.

Problem 6. Two vertices of a trapezoid are at $(-2,0)$ and $(2,0)$, and the other two lie on the semicircle $x^2+y^2=4$, $y \geq 0$. What is the maximum possible area of the trapezoid? (Recall that the area of a trapezoid with bases b_1 and b_2 and height h is $h(b_1+b_2)/2$).

Problem 7. The Lanczos derivative of a function $f(x)$ at a point a is defined as

$$f_L^{'}(a) := \lim_{h \mapsto 0^+} \frac{3}{2h^3} \int_{-h}^{h} t f(a+t) dt$$

provided the limit exists.

(a) Let $a \in \mathbb{R}$. Compute $f_L^{'}(a)$ when $f(x)=x$ and when $f(x)=x^2$.
(b) Compute $f_L^{'}(0)$ when $f(x)=|x|$. How does this compare to $f^{'}(0)$?
(c) Does the mean value theorem hold for $f(x)=|x|$ on the interval $[-1,3]$ if the usual derivative is replaced by the Lanczos derivative?

Problem 8. Do the following series converge or diverge? Completely justify your answer.

(a)

$$\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n}).$$

(b)

$$\sum_{n=1}^{\infty}(\sqrt[n]{n}-1)^n.$$

Problem 9. Let $F(x,y)=\int_0^x e^{-5t-3y}dt - \int_{x/2}^{y} e^{-3t}dt$.

(a) Find a linear approximation of F at $(0,1/3)$.
(b) Identify the points on the unit circle where this linear approximation is maximized and where it is minimized. It is not necessary to obtain a simplified expression for these points and you do not have to specify which is the maximizer or minimizer.

Problem 10. Let $f(x)=x^{\frac{-x}{x-1}}$ be defined on $(0,1)$.

(a) Use $\log z \leq z-1$ to show $f(x) \geq e^{-1}$ on $(0,1)$.
(b) Compute $\lim_{x \to 1^-} f(x)$.