Graduate Qualifying Exam – Solutions – Spring 2022

Note: You have *three hours* to complete this exam. Calculators and other electronic devices are *not allowed*. Show all work to receive full credit.

MATH 124

Problem 1. Let $f(x) = e^{-|x|}$, $x \in \mathbb{R}$.
(a) Use the definition of the derivative (directly) to decide whether or not f is differentiable at $x = 0$ and justify your decision.
(b) Consider a rectangle whose vertices are given by (x, y), $(-x, y)$, $(x, 0)$, and $(-x, 0)$, $x > 0$, $y = f(x)$. Find the maximum area of the rectangle.

Solution. (a) Note that

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{e^{-h} - 1}{h} = \lim_{h\to 0^+} -e^{-h} = -1$$

and

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{e^{h} - 1}{h} = \lim_{h\to 0^-} e^{h} = 1.$$

Thus, these two limits do not agree, implying that the $f(x)$ is not differentiable at $x = 0$. The limits are evaluated using L'Hopital's Rule.
(b) We want to maximize $g(x, y) = 2xy$ subject to $y = e^{-|x|}$. However, because this rectangle is symmetric around the y-axis, we only need to consider the right half of the rectangle first. In other words, we want to maximize xy subject to $y = e^{-x}$, $x > 0$. This is the same as maximizing $h(x) = xe^{-x}$. Because $g'(x) = e^{-x}(1 - x) = 0$ at $x = 1$, and $h''(x) = -e^{-x}(2 - x) = -e^{-1} < 0$ at $x = 1$, h is maximized at $x = 1$, which gives $g(1, e^{-1}) = 2e^{-1} \approx 0.7358$ as the maximum area of the rectangle.

MATH 125

Problem 2. Consider the function $f : [0, \sqrt{2\pi}] \to \mathbb{R}$ given by

$$f(x) = \int_0^{x^2} \frac{\sin t}{1 + t}\, dt.$$

(a) Compute the exact value of $f'\left(\sqrt{\frac{3\pi}{2}}\right)$.
(b) Is f a decreasing function on its domain? Justify your answer.
(c) Compute the exact value of $\displaystyle\int_0^{\sqrt{\pi}} (1 + x^2) f'(x)\, dx$.

Solution. (a) Note that $f(x) = F(x^2)$, where $F(x) = \int_0^x \frac{\sin t}{1+t}\, dt$. Thus, f is differentiable and $f'(x) = 2xF'(x^2) = \frac{2x\sin(x^2)}{1+x^2}$. We compute the exact value requested as $-\frac{2\sqrt{6\pi}}{2+3\pi}$.
(b) No, since the derivative is not always negative on the domain.

(c) Note that $(1+x^2)f'(x) = 2x\sin(x^2)$. This is a typical example of substitution. We get that the indefinite integral is $-\cos(x^2) + C$, thus the definite integral evaluates to 2.

MATH 224

Problem 3. A caravan is trying to cross the desert while minimizing a certain distance metric. Currently, the caravan is at $(5,3)$ and its final destination is at $(2,-7)$ on the Cartesian plane. However, the camels will not survive unless the caravan makes a stop at the oasis which has a round shape with radius $r = 1$ centered at the origin. The distance metric they are minimizing is the sum of squared distances $[D_1(x,y)]^2 + [D_2(x,y)]^2$. Here, $D_1(x,y)$ is the Euclidean distance between the current location and (x,y), the location at which the caravan will make the stop. Similarly, $D_2(x,y)$ is the Euclidean distance between (x,y) and the final destination. Determine (x,y), the location at which the caravan will make the stop, by setting up an appropriate function to be minimized and the constraint explicitly.

Solution. Function to be minimized:

$$f(x,y) = (x-5)^2 + (y-3)^2 + (x-2)^2 + (y+7)^2.$$

Constraint:

$$g(x,y) = x^2 + y^2 - 1 = 0.$$

Let $h(x,y,\lambda) = f(x,y) - \lambda g(x,y)$. Then,

$$\frac{\partial h}{\partial x} = 2(2-\lambda)x - 14 \text{ and } \frac{\partial h}{\partial y} = 2(2-\lambda)y + 8.$$

Therefore, by setting $\frac{\partial h}{\partial x} = 0$ and $\frac{\partial h}{\partial y} = 0$, we have $x = \frac{7}{2-\lambda}$ and $y = -\frac{4}{2-\lambda}$. Plugging these into the constraint and solving for λ gives $\lambda = 2 \pm \sqrt{65}$ so that the possible solutions are:

$$(x,y) = \left(\frac{7}{\sqrt{65}}, -\frac{4}{\sqrt{65}}\right) \text{ or } (x,y) = \left(-\frac{7}{\sqrt{65}}, \frac{4}{\sqrt{65}}\right).$$

Using the objective function, we find that the location at which the caravan will make the stop is $(x,y) = \left(\frac{7}{\sqrt{65}}, -\frac{4}{\sqrt{65}}\right) \approx (0.8682, -0.4961)$.

MATH 225

Problem 4. Let D be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by $x^2 + y^2 + z^2 = z$. Find the volume of D by setting up an appropriate triple integral and then computing its value.

Solution. This problem can be solved by the cylindrical or spherical coordinates. Below is a solution using the cylindrical coordinates.
Since

$$x^2 + y^2 + (z-0.5)^2 = 0.5^2,$$

the two equations intersect at $z = 0.5$, giving $0 \leq r \leq 0.5$, $0 \leq \theta \leq 2\pi$, and $r \leq z \leq \sqrt{0.25 - r^2} + 0.5$. Therefore, the volume of D, denoted by V, is given by

$$\begin{aligned} V &= \int_0^{0.5} \int_0^{2\pi} \int_r^{\sqrt{0.25-r^2}+0.5} r dz d\theta dr \\ &\dots \\ &= 2\pi \int_0^{0.5} r(\sqrt{0.25 - r^2} + 0.5) - r^2 dr \\ &= \pi \left[\int_0^{0.5} 2r\sqrt{0.25 - r^2} dr + \int_0^{0.5} r - 2r^2 dr \right] \\ &= \pi \int_0^{0.25} \sqrt{u} dr + \pi \int_0^{0.5} r - 2r^2 dr \\ &= \pi \left(\frac{1}{12} + \frac{1}{24} \right) \\ &= 0.125\pi \approx 0.3927 \end{aligned}$$

by setting $u = 0.25 - r^2$.

MATH 226

Problem 5. (a) Find the interval of convergence of the power series $\sum_{n \geq 1} \frac{x^n}{n(n+1)}$.

(b) For x in the interval of convergence found in part (a), let

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}.$$

Compute the exact value of $S(1)$.

Solution. (a) For example, by Ratio Test the series converges absolutely for $x \in (-1, 1)$ and diverges on $\mathbb{R} \setminus [-1, 1]$. At the end points: if $x = 1$, the series converges-say by comparison with 2-series, while at $x = -1$ the series converges by the Alternating Series Test. The interval of convergence is $[-1, 1]$.

(b) Let $S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$. Then

$$S(1) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

We note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Thus we recognize this as a telescoping series and $S(1) = 1$.

Problem 6. The sequence $(s_n)_{n\geq 1}$ is defined recursively by $s_1 = 1$ and

$$s_{n+1} = \sqrt{1+s_n}, n \geq 1.$$

(a) Show that $(s_n)_{n\geq 1}$ is a monotonic sequence.
(b) Show that $(s_n)_{n\geq 1}$ is a bounded sequence.
(c) Based on parts (a)-(b), there exists $\ell = \lim_{n\to\infty} s_n$. Compute the exact value of ℓ.
(d) Prove that the series $\sum_{n\geq 0} \ell^{-n}$ converges and find the exact value of its sum.

Solution. (a)-(b) By induction, one can prove that $0 < s_n < 2$ for all $n \in \mathbb{N}$ and that (s_n) is strictly increasing.
(c) The Monotonic Convergence Theorem guarantees the existence of ℓ. From the recursion given we find that $\ell = \sqrt{1+\ell}$, that is $\ell^2 - \ell - 1 = 0$. Out of the possible two roots we select the positive one, $\ell = \frac{1+\sqrt{5}}{2}$ (incidentally, the golden ratio).
(d) This is a geometric series with ratio $\ell^{-1} = \frac{-1+\sqrt{5}}{2} < 1$. Thus it converges to

$$\frac{1}{1-\frac{-1+\sqrt{5}}{2}} = \frac{2}{3-\sqrt{5}} = \frac{3+\sqrt{5}}{2}.$$

MATH 204/304

Problem 7. (a) Show that if A is a diagonalizable matrix with non-negative real eigenvalues, then there is a matrix S such that $S^2 = A$.
(b) Find the matrix S that satisfies $S^2 = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$.

Solution. If the $n \times n$ matrix A is diagonalizable, then $A = PDP^{-1}$, where D is a diagonal matrix containing the eigenvalues $\lambda_j, 1 \leq j \leq n$ on its main diagonal, and the columns of P made of the corresponding eigenvectors. Simply let $S = P\sqrt{D}P^{-1}$, where $\sqrt{D}$ is the diagonal matrix with entries $\sqrt{\lambda_j}$ on its main diagonal (in the same order as in D). Then $S^2 = P(\sqrt{D})^2P^{-1} = PDP^{-1} = A$.
(b) The matrix A in this case is very simple (almost diagonal). So, there is a simple guess one can make:

$$S = \begin{bmatrix} 1 & 0 & x \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

It follows immediately that $x = 2$ works. Alternately, following up on part (a): the eigenvalues are $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 9$. We already have D and, hence $\sqrt{D}$. Since the eigenvalues are distinct, we know that A is diagonalizable. To find P, we find corresponding eigenvectors. For example,

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -8 \\ 0 \\ 1 \end{bmatrix}.$$

Thus our desired matrix is $P = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ and S as defined in part (a).

Problem 8. Suppose that the weather in a particular region of the globe behaves according to the following scenario: the probability that tomorrow will be a wet day is 0.662 if today is wet and 0.25 if today is dry, while the probability that tomorrow will be a dry day is 0.75 if today is dry and 0.338 if today is wet.
(a) Find the matrix P such that

$$\begin{bmatrix} \text{Wet} \\ \text{Dry} \end{bmatrix}_{\text{[tomorrow]}} = P \begin{bmatrix} \text{Wet} \\ \text{Dry} \end{bmatrix}_{\text{[today]}}$$

(b) If Friday is a dry day, what is the probability that Sunday will be wet? Explain in terms of the matrix P from part (a).
(c) What will be the distribution of wet and dry days in the long run?

Solution. (a) The top row of the matrix P captures the transition of wet days, while the bottom row captures the transition of dry days:

$$P = \begin{bmatrix} .662 & .25 \\ .338 & .75 \end{bmatrix}$$

(b) The vector that captures the weather on Friday is $\vec{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The vector that captures the weather on Sunday is $P^2\vec{v}_0 = \begin{bmatrix} .353 \\ .647 \end{bmatrix}$.
(c) One can find the eigenvalues of P. The important eigenvalue is $\lambda = 1$ (since in the limit, the other eigenvalue will yield the zero vector). So, we are interested in the appropriate eigenvector $\vec{v}_1$ (the entries should be positive numbers adding up to 1 as they represent probabilities). We find that (up to three decimal places), $\vec{v}_1 = \begin{bmatrix} .425 \\ .575 \end{bmatrix}$, so .425 probability wet and .575 probability dry.

Problem 9. Let A be a square matrix and $\vec{v}, \vec{w}$ be two non-zero vectors such that

$$A\vec{v} = 2022\vec{w} \text{ and } A\vec{w} = 2022\vec{v}. \quad (*)$$

(a) Prove that 2022 or -2022 is an eigenvalue of A.
(b) Give an example of a 2×2 matrix A that is not invertible and satisfies $(*)$.

Solution. (a) Note that if $\vec{v} = \vec{w}$, 2022 is an eigenvalue of A. Assume that $\vec{v} \neq \vec{w}$. Then subtracting the two equations gives $A(\vec{v} - \vec{w}) = -2022(\vec{v} - \vec{w})$. Since $\vec{v} - \vec{w} \neq \vec{0}$, this shows that -2022 is an eigenvalue of A.
(b) A diagonal matrix with entries 2022 and 0 on the main diagonal.

MATH 331

Problem 10. An object of mass $m = 9$ kg is attached to a spring with unknown spring constant $k > 0$. There is no damping present. Let $x(t)$ be the distance of the object from the equilibrium position at time t. The mass was initially displaced 1 m from its equilibrium position and released without any initial velocity. Assuming that it took 3 seconds for the object to reach the equilibrium for the first time, find the exact least positive value of k.

Solution. The differential equation governing the motion is $9x'' + kx = 0$ or $x'' + \omega^2 x = 0$, where $\omega = \frac{\sqrt{k}}{3}$. Therefore, $x(t) = a\cos(\omega t) + b\sin(\omega t)$, for some constants $a, b \in \mathbb{R}$. Since $x(0) = 1$ and $x'(0) = 0$, we find that $a = 1$ and $b = 0$. The solution is

$$x(t) = \cos(\frac{\sqrt{k}}{3}t).$$

Now, given that $x(3) = 0$ we further find that $\cos(\sqrt{k}) = 0$ which yields $k = \frac{\pi^2}{4}$.
