Western Washington University
Department of Mathematics

Fall 2022 Graduate Qualifying Exam Solutions

1. Let

$$f(x) = \begin{cases} \frac{x^3-x^2}{x-1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1. \end{cases}$$

   (a) Does $\lim_{x\to 1} f(x)$ exist? Either state and prove the limit using an epsilon-delta argument, or formally justify why it does not exist.

   (b) Is $f(x)$ differentiable at $x = 1$? Justify your answer using the definition of the derivative.

   [Solution] For (a), we'll show that $\lim_{x\to 1} f(x) = 1$. Let $\epsilon > 0$ and choose $\delta = \min\{1, \frac{\epsilon}{3}\}$. Then for all x with $0 < |x-1| < \delta$, we have $|x+1| < 3$, so

$$\left| \frac{x^3 - x^2}{x-1} - 1 \right| = |x^2 - 1| = |x+1||x-1| < \epsilon.$$

   For (b), we have

$$\lim_{x\to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1} \frac{x^2}{x-1} = \infty.$$

   Since the limit is not finite, $f'(1)$ does not exist.

2. Let

$$f(x,y) = \int_x^{\sqrt{y+\frac{\pi}{4}}} \cos\left(t^2\right)\, dt.$$

   Estimate the change in $f(x,y)$ from $(0,0)$ to $(0.1, 0.04)$ using the linear approximation at $(0,0)$.

   [Solution] $\nabla f(0,0) = (f_x(0,0), f_y(0,0)) = \left(-\cos 0, \frac{\cos\frac{\pi}{4}}{2\sqrt{\frac{\pi}{4}}} \right) = \left(-1, \frac{1}{\sqrt{2\pi}} \right)$. Then

$$f(0.1, 0.04) - f(0,0) \approx -\Delta x + \frac{1}{\sqrt{2\pi}}\Delta y = -0.1 + \frac{1}{\sqrt{2\pi}}(0.04).$$

3. Suppose that the linear system $A\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ has the general solution

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{with free variables } c,\ d.$$

(a) Find a basis for the null space of A.

(b) Find a basis for the column space of A.

(c) Determine the matrix A.

[**Solution**] (a) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for Nul A.

(b) Since A is 3×3 and $\dim(\text{Nul } A) = 2$, $\dim(\text{Col } A) = 1$, so $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ is a basis for Col A.

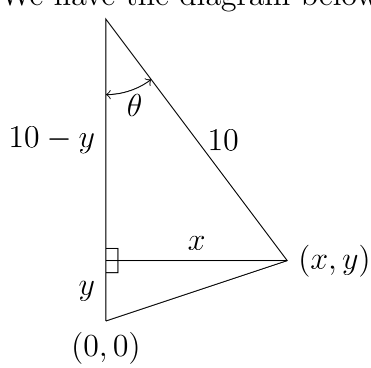
(c) $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ since $A \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$.

4. A swing consists of a board at the end of a 10 foot long rope that is attached to a tree branch at the other end. Suppose that the swing is directly below the point of attachment at time $t = 0$, and is being pushed by someone who walks at 6 feet per second in the horizontal direction.

(a) How fast is the swing rising after 1 second?

(b) What is the angular speed of the rope in radians per second after 1 second?

[**Solution**] Let $(0, 0)$ be the position of the swing at $t = 0$ and (x, y) its position at time t. We have the diagram below, and are given that $\frac{dx}{dt} = 6$ ft/sec.



For (a), we want $\frac{dy}{dt}$. We implicitly differentiate $x^2 + (10 - y)^2 = 100$ to get $2x\frac{dx}{dt} - 2(10 - y)\frac{dy}{dt} = 0$. At $t = 1$, $x = 6$ and $y = 2$, so $\frac{dx}{dt} = \frac{9}{2}$ ft/sec.

For (b), we have $\sin\theta = \frac{x}{10}$ so $\cos\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$. At $t = 1$, $\cos\theta = \frac{4}{5}$ hence $\frac{d\theta}{dt} = \frac{3}{4}$ radians per second.

5. Let $T : V \to V$ be a linear transformation such that $T \circ T = T$.

   (a) Give an example of such a function with $V = \mathbb{R}^2$ such that T is neither the zero map nor the identity map.

   (b) Show that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly dependent if and only if $T(\mathbf{v}) = \mathbf{v}$ or $T(\mathbf{v}) = \mathbf{0}$.

   [**Solution**] (a) T is represented by a 2×2 matrix A with $A^2 = A$. One such matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, which corresponds geometrically to projection onto the horizontal axis.

   (b) If the set is either $(\mathbf{v}, \mathbf{0})$ or $(\mathbf{v}, \mathbf{v})$ it is clearly dependent. Conversely, let $\{\mathbf{v}, T(\mathbf{v})\}$ be linearly dependent. Then $a\mathbf{v} + bT(\mathbf{v}) = \mathbf{0}$ for scalars a and b not both zero. Applying T to the equation, we get $(a+b)T(\mathbf{v}) = \mathbf{0}$, so either $T(\mathbf{v}) = \mathbf{0}$, or $a + b = 0$. In the second case, $b = -a$ with $a \neq 0$ so $a\mathbf{v} = aT(\mathbf{v})$ and hence $T(\mathbf{v}) = \mathbf{v}$.

6. What fraction of the volume of a sphere is contained between two parallel planes that trisect the diameter to which they are perpendicular?

   [**Solution**] Without loss of generality, consider the unit sphere centered at the origin. Let V be the volume of the sphere and let V_1 be the volume of the upper removed section.

$$\begin{aligned} \frac{V - 2V_1}{V} &= 1 - \frac{2\int_0^{2\pi}\int_0^{2\sqrt{2}/3}\int_{1/3}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta}{4\pi/3} \\ &= 1 - \frac{1}{3}\left[-\frac{1}{3}(1 - r^2)^{3/2} - \frac{1}{6}r^2\right]_0^{2\sqrt{2}/3} \\ &= \frac{13}{27}. \end{aligned}$$

7. Determine whether each series below is absolutely convergent, conditionally convergent, or divergent. Explain your reasoning by citing appropriate tests.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)!}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

[Solution] (a) This series converges absolutely by the ratio test. (b) $f(x) = \frac{1}{x \ln x}$ is continuous and positive on $[2, \infty)$. Since $f'(x) = \frac{-1-\ln x}{(x \ln x)^2} < 0$, f is decreasing on $[2, \infty)$, and $\displaystyle\int_2^\infty f(x)\, dx$ diverges, so by the integral test, the series does not converge absolutely. Since $\frac{1}{n \ln n} \to 0$ as $n \to \infty$, and the terms are decreasing, then the series is conditionally convergent by the alternating series test.

8. Find the general solution to the partially decoupled linear system

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = x + y - 2.$$

[Solution] Solve the 1st equation for the general solution, $x(t) = k_1 e^t$. Then the 2nd equation becomes $\dfrac{dy}{dt} = y + k_1 e^t - 2$, which can be solved using extended linearity principle.

Guess a particular solution $y_{p1}(t) = \alpha t e^t$ to $\dfrac{dy}{dt} = y + k_1 e^t$, plugging in and determine $\alpha = k_1$. In addition, $y_{p2}(t) = 2$ is a particular solution to $\dfrac{dy}{dt} = y - 2$. Therefore, $y_{p1}(t) + y_{p2}(t) = k_1 t e^t + 2$ is a particular solution to $\dfrac{dy}{dt} = y + k_1 e^t - 2$.

Since $y_h(t) = e^t$, by the extended linearity principle, the general solution to the 2nd equation is

$$y(t) = k_2 e^t + k_1 e^{2t} - 1.$$

Therefore, the general solution to the nonlinear system is

$$(x(t), y(t)) = (k_1 e^{2t}, k_2 e^t + k_1 t e^t + 2).$$

9. Let S be the graph of $f(x, y) = \dfrac{1}{xy}$ and Π be the tangent plane of S through the point $(-1, -1, 1)$. A particle P at the point $(1/2, 1, 2)$ on S, following the surface normal direction there, heads toward Π along a straight path. If θ is the acute angle between the particle's path and Π, find $\sin \theta$.

[Solution] Take $\mathbf{n_1} = \langle 4, 2, 1 \rangle$ as the normal at $(1/2, 1, 2)$ and $\mathbf{n_2} = \langle 1, 1, -1 \rangle$ the normal of Π, so

$$\sin \theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{\|\mathbf{n_1}\| \|\mathbf{n_2}\|} = \frac{5}{\sqrt{63}}.$$

10. An open rectangular box (a box without a top) of volume 80 cm^3 is to be constructed from material that costs \$5/cm^2 for the bottom and \$2/cm^2 for its sides. What dimensions should the box have in order to minimize the total cost?

[**Solution**] Assume the optimal dimensions are $l \times w \times h$.

$$\min \quad C(l,w,h) = 5lw + 2 \cdot 2(lh + hw)$$
$$\text{subject to} \quad V(l,w,h) = lwh = 80$$

By method of Lagrange multipliers,

$$\nabla C = \lambda \nabla V \quad \Rightarrow \quad 5w + 4h = \lambda \cdot wh, \quad 5l + 4h = \lambda \cdot lh, \quad 4(l+w) = \lambda \cdot lw$$

Dividing the first two equations yields

$$\frac{5w+4h}{5l+4h} = \frac{w}{l} \quad \Rightarrow \quad w = l.$$

Dividing the 2nd and 3rd equations yields

$$\frac{5l+4h}{4(l+w)} = \frac{h}{w} \quad \Rightarrow \quad w = \frac{4}{5}h.$$

Plugging in the constraint,

$$V(l,w,h) = lwh = \frac{4}{5}h \cdot \frac{4}{5}h \cdot h = 80 \quad \Rightarrow \quad h = 5 \quad \Rightarrow \quad l = w = \frac{4}{5}h = 4.$$

So the optimal dimensions are $4 \times 4 \times 5$.