Western Washington University
Department of Mathematics

Fall 2022 Graduate Qualifying Exam

September 12, 2022

Instructions: This exam begins at 9 am and ends at 12 pm. Each problem is worth 10 points. You may use a graphing calculator, but you must show all your work to receive full credit. Your answers should be exact values, not approximations. Clearly identify the problem number for each of your solutions. Write your secret ID number (not your name) on each page.

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1. Let

$$f(x) = \begin{cases} \frac{x^3 - x^2}{x-1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1. \end{cases}$$

   (a) Does $\lim_{x \to 1} f(x)$ exist? Either state and prove the limit using an epsilon-delta argument, or formally justify why it does not exist.

   (b) Is $f(x)$ differentiable at $x = 1$? Justify your answer using the definition of the derivative.

2. Let

$$f(x, y) = \int_x^{\sqrt{y + \frac{\pi}{4}}} \cos\left(t^2\right)\, dt.$$

   Estimate the change in $f(x, y)$ from $(0, 0)$ to $(0.1, 0.04)$ using the linear approximation at $(0, 0)$.

3. Suppose that the linear system $A\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ has the general solution

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{with free variables } c, d.$$

   (a) Find a basis for the null space of A.

   (b) Find a basis for the column space of A.

   (c) Determine the matrix A.

4. A swing consists of a board at the end of a 10 foot long rope that is attached to a tree branch at the other end. Suppose that the swing is directly below the point of attachment at time $t = 0$, and is being pushed by someone who walks at 6 feet per second in the horizontal direction.

   (a) How fast is the swing rising after 1 second?

   (b) What is the angular speed of the rope in radians per second after 1 second?

5. Let $T : V \to V$ be a linear transformation such that $T \circ T = T$.

   (a) Give an example of such a function with $V = \mathbb{R}^2$ such that T is neither the zero map nor the identity map.

   (b) Show that $\{\mathbf{v}, T(\mathbf{v})\}$ is linearly dependent if and only if $T(\mathbf{v}) = \mathbf{v}$ or $T(\mathbf{v}) = \mathbf{0}$.

6. What fraction of the volume of a sphere is contained between two parallel planes that trisect the diameter to which they are perpendicular?

7. Determine whether each series below is absolutely convergent, conditionally convergent, or divergent. Explain your reasoning by citing appropriate tests.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)!}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

8. Find the general solution to the partially decoupled linear system
$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = x + y - 2.$$

9. Let S be the graph of $f(x, y) = \dfrac{1}{xy}$ and Π be the tangent plane of S through the point $(-1, -1, 1)$. A particle P at the point $(1/2, 1, 2)$ on S, following the surface normal direction there, heads toward Π along a straight path. If θ is the acute angle between the particle's path and Π, find $\sin\theta$.

10. An open rectangular box (a box without a top) of volume 80 cm^3 is to be constructed from material that costs \$5/cm^2 for the bottom and \$2/cm^2 for its sides. What dimensions should the box have in order to minimize the total cost?