Graduate Qualifying Exam – Spring 2022

March 28, 2022

Note: You have *three hours* to complete this exam. Calculators and other electronic devices are *not allowed*. Show all work to receive full credit.

Problem 1. Let $f(x) = e^{-|x|}$, $x \in \mathbb{R}$.
(a) Use the definition of the derivative (directly) to decide whether or not f is differentiable at $x = 0$ and justify your decision.
(b) Consider a rectangle whose vertices are given by (x, y), $(-x, y)$, $(x, 0)$, and $(-x, 0)$, $x > 0$, $y = f(x)$. Find the maximum area of the rectangle.

Problem 2. Consider the function $f : [0, \sqrt{2\pi}] \to \mathbb{R}$ given by

$$f(x) = \int_0^{x^2} \frac{\sin t}{1+t}\, dt.$$

(a) Compute the exact value of $f'\left(\sqrt{\frac{3\pi}{2}}\right)$.
(b) Is f a decreasing function on its domain? Justify your answer.
(c) Compute the exact value of $\displaystyle\int_0^{\sqrt{\pi}} (1+x^2) f'(x)\, dx$.

Problem 3. A caravan is trying to cross the desert while minimizing a certain distance metric. Currently, the caravan is at $(5, 3)$ and its final destination is at $(2, -7)$ on the Cartesian plane. However, the camels will not survive unless the caravan makes a stop at the oasis which has a round shape with radius $r = 1$ centered at the origin. The distance metric they are minimizing is the sum of squared distances $[D_1(x, y)]^2 + [D_2(x, y)]^2$. Here, $D_1(x, y)$ is the Euclidean distance between the current location and (x, y), the location at which the caravan will make the stop. Similarly, $D_2(x, y)$ is the Euclidean distance between (x, y) and the final destination. Determine (x, y), the location at which the caravan will make the stop, by setting up an appropriate function to be minimized and the constraint explicitly.

Problem 4. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by $x^2 + y^2 + z^2 = z$. Find the volume of D by setting up an appropriate triple integral and then computing its value.

Problem 5. (a) Find the interval of convergence of the power series $\sum_{n\geq 1} \frac{x^n}{n(n+1)}$.
(b) For x in the interval of convergence found in part (a), let

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}.$$

Compute the exact value of $S(1)$.

Problem 6. The sequence $(s_n)_{n\geq 1}$ is defined recursively by $s_1 = 1$ and

$$s_{n+1} = \sqrt{1+s_n}, \quad n \geq 1.$$

(a) Show that $(s_n)_{n\geq 1}$ is a monotonic sequence.
(b) Show that $(s_n)_{n\geq 1}$ is a bounded sequence.
(c) Based on parts (a)–(b), there exists $\ell = \lim_{n\to\infty} s_n$. Compute the exact value of ℓ.
(d) Prove that the series $\sum_{n\geq 0} \ell^{-n}$ converges and find the exact value of its sum.

Problem 7. (a) Show that if A is a diagonalizable matrix with non-negative real eigenvalues, then there is a matrix S such that $S^2 = A$.
(b) Find the matrix S that satisfies $S^2 = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$.

Problem 8. Suppose that the weather in a particular region of the globe behaves according to the following scenario: the probability that tomorrow will be a wet day is 0.662 if today is wet and 0.25 if today is dry, while the probability that tomorrow will be a dry day is 0.75 if today is dry and 0.338 if today is wet.
(a) Find the matrix P such that

$$\begin{bmatrix} \text{Wet} \\ \text{Dry} \end{bmatrix}_{[\text{tomorrow}]} = P \begin{bmatrix} \text{Wet} \\ \text{Dry} \end{bmatrix}_{[\text{today}]}$$

(b) If Friday is a dry day, what is the probability that Sunday will be wet? Explain in terms of the matrix P from part (a).
(c) What will be the distribution of wet and dry days in the long run?

Problem 9. Let A be a square matrix and $\vec{v}, \vec{w}$ be two non-zero vectors such that

$$A\vec{v} = 2022\vec{w} \text{ and } A\vec{w} = 2022\vec{v}. \quad (*)$$

(a) Prove that 2022 or -2022 is an eigenvalue of A.
(b) Give an example of a 2×2 matrix A that is not invertible and satisfies $(*)$.

Problem 10. An object of mass $m = 9$ kg is attached to a spring with unknown spring constant $k > 0$. There is no damping present. Let $x(t)$ be the distance of the object from the equilibrium position at time t. The mass was initially displaced 1 m from its equilibrium position and released without any initial velocity. Assuming that it took 3 seconds for the object to reach the equilibrium for the first time, find the exact least positive value of k.
