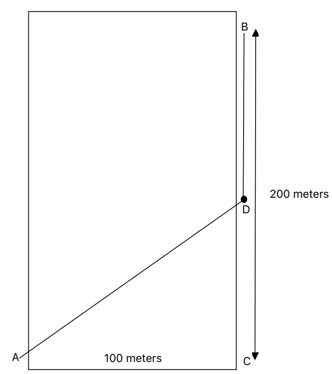
Qualifying Exam, Fall 2021

This exam starts at 9 a.m. and ends at 1 p.m. Calculators may **not** be used. Please provide complete answers that show your work. Clearly identify each problem you are working on on your page. Write your secret ID on **every page of your exam**, including scratch work.

Problem 1. (10 points) A pipe will cross from point A to point B on the two sides of a river (see the figure), passing through the point D (which is to be found). The river has parallel banks and its (constant) width is 100m. The point C is directly across the river from the point A and the distance from C to B is 200m. The pipe costs \$1000 per meter on land (between D and B) and \$1500 per meter over the river (between A and D).

Where should D be placed to minimize the cost of the pipe (in this portion between A and B)? Your work should clearly define variables and describe the steps taken as well as reporting the location of D.



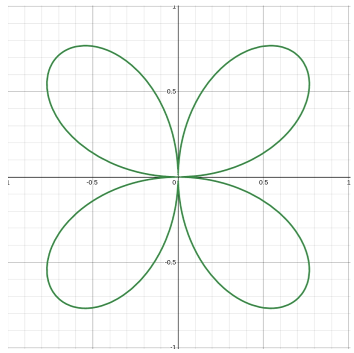
Problem 2. This problem has three parts.
(a) (4 points) Clearly state the mean value theorem, relating the mean value and derivative of a function.
(b) (3 points) Does there exist a differentiable function $f : \mathbf{R} \to \mathbf{R}$ with $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all x? Prove your answer.
(c) (3 points) Does there exist a differentiable function $f : \mathbf{R} \to \mathbf{R}$ with $f(0) = -1$, $f(2) = 4$ and $f'(x) \geq 2$ for all x? Prove your answer.

Problem 3. For $t \geq 1$, consider the function

$$f(t) = \int_1^2 \frac{1+5t}{1+t\,x^2}\,dx + \int_1^t \frac{3}{1+x^3}\,dx.$$

(a) (6 points) Show that $f(t)$ is strictly increasing over its domain.
(b) (4 points) Is $f(t)$ bounded for $t \geq 1$? Explain.

Problem 4. This problem deals with the (four petaled) rose, which is the curve shown below.



This curve has parameterization

$$x = \sin(2t)\cos(t), \quad y = \sin(2t)\sin(t), \qquad t \in \mathbf{R}.$$

(a) (3 points) Show that $(x(t)^2 + y(t)^2)^3 = 4x(t)^2 y(t)^2$. (You need not prove this, but any point (x, y) satisfying $(x^2 + y^2)^3 = 4x^2y^2$ must lay on the rose.)
(b) (4 points) Compute the tangent line to the curve at the point $(1/\sqrt{2}, 1/\sqrt{2})$. No eye-balling it.
(c) (3 points) Set up, but do not compute or simplify, an integeral that computes the arc length of one of the petals. Your integral should be in terms of the usual trigonometric functions and arithmetic operations.

Problem 5. Consider the following three sums.
(a) (3 points) Evaluate: $\sum_{n=0}^{\infty} \frac{1}{3^n}$.
(b) (3 points) Evaluate: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{6}\right)^{2n+1}$.
(c) (4 points) Does $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ converge? (Hint: $\frac{d}{dx} \ln(x) = ?$)

Problem 6. (10 points) The matrix A is 2×3, and its image contains all of $\mathbf{R}^2$ (the transformation is onto, or surjective). You are given that

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Describe all solutions, $\mathbf{x}$, to $A\mathbf{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. Explain how you know that you have found all the solutions.

Problem 7. Ler R be the region $R = \{(x, y) \,|\, x \geq 0,\ y \geq 0,\ x + 2y \leq 4\}$. Set $f(x, y) = x^2 - 2x - y + y^2$ and set $g(x, y) = xy$.
(a) (5 points) Find the maximum value of g on R and the point(s) at which this maximum occurs.
(b) (5 points) Find the minimum value of f on R and the point(s) at which this value occurs.

Problem 8. (a) (5 points) Evaluate the integral of $f(x,y) = e^{x^2}$ over the region $T = \{(x,y)\,|\,0 \le y \le 1,\ y \le x \le 1\}$.
(b) (5 points) Evaluate the integral of $g(x,y) = x$ over the region $D = \{(x,y)\,|\,x^2 + y^2 \le 4,\ x \ge 1\}$.

Problem 9. Let A be a **real** 2-by-2 matrix satisfying $A^4 = 16 \cdot I$ (where I is, of course, the identity matrix).

(a) (3 points) What are the possible (complex) eignenvalues for A?

(b) (4 points) For each eigenvalue λ you listed above, give an example of a **real** 2-by-2 matrix A with λ as an eigenvalue so that matrix satisfies $A^4 = 16 \cdot I$.

(c) (3 points) Determine the **real** eigenvectors of the matrices you gave in (b), or explain why no real eigenvectors exist.

Problem 10. Consider the differential equation $x'' + 2px' + x = 3$, where $x = x(t)$ and p is a real constant satisfying $|p| < 1$.
(a) (2 points) Show that there is a constant solution $x(t) \equiv c$ to the above equation.
(b) (8 points) Prove that every solution $x(t)$ converges to the constant solution $\lim_{t\to\infty} x(t) = c$.