Calculators are allowed but you must give exact values, not approximate answers. You must clearly justify all answers. Problems are of equal weight.

Number: ______________________________

1. Find the value of a for which the integral

$$\int_1^\infty \frac{a}{x(2x+a)}dx$$

converges to the value of 1.

2. a. Show, by an example, that linear dependence of the columns of a matrix does not imply the linear dependence of the rows. (Note: you need to *briefly* indicate why the columns of your matrix are linearly dependent and the rows are not. A proof is not needed.)

   b. State the rank-nullity theorem, a.k.a. the dimension theorem, for matrices.

   c. Show that, if an $n \times n$ matrix A is such that $A^2 = A$, then $\text{rank}(A) + \text{rank}(I - A) = n$.

3. Find the real-valued function function $y : \mathbb{R} \to \mathbb{R}$ which satisfies

$$\begin{aligned} 2y'' - y' - 6y &= 0, \\ y(0) &= 1, \\ y'(0) &= 0. \end{aligned}$$

4. Let M denote the $n \times n$ matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Note that $M^2 = nM$.

   a. What are the possible eigenvalues of $aI_n + bM$? Here a and b are scalars and I_n is the identity $n \times n$ matrix.

   b. Let A be the 4×4 matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

   Let k be an integer greater than 1. Find A^k. Note: You may express your answer in the form of the product $B^x H C^y$, where B, H, C are nontrivial matrices whose elements must be specified.

5. Let $f_n(x)$ be defined recursively as $f_n(x) = 1 - \int_0^x s[f_{n-1}(s)]ds$ with $f_0(x) = 1$.

   a. Determine $f_1(x), f_2(x), f_3(x), f_4(x)$.

   b. What is the coefficient of the highest order term in f_n?

   c. Show that for each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x)$ exists.

6. A curve is given by the parametric equations

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 < t < \frac{\pi}{2}.$$

   a. Show that, near $t = 0$, the equations can be approximated by

$$a - x \approx 3at^2/2, \quad y \approx at^3.$$

   b. Find the length of the curve. *Note:* A useful identity is $\sin 2\theta = 2\sin\theta\cos\theta$.

7. A company is looking to invest in two assets. If w_1 and w_2 are the amounts of the investment budget invested in each asset, then the variance of the total investments is given by

$$\frac{1}{4}w_1^2 + \frac{1}{9}w_2^2 + \frac{1}{3}\rho w_1 w_2$$

   where $\rho \in [-1, 1]$ is a known fixed number. The expected return of this investment is given by the expression

$$8w_1 + 4w_2$$

   What is the best choice of w_1 and w_2—in terms of ρ—to minimize the variance while ensuring an expected return of 24?

8. a. Prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\lim_{n\to\infty} n^\alpha a_n = A$ for some $\alpha > 1$, where A is some real number.

   b. For each of the following, determine whether the series converges or diverges and provide the justification of your answer.

      (ii) $\sum_{n=1}^{\infty} \sin(\frac{\pi}{n^2})$

      (ii) $\sum_{n=1}^{\infty} \frac{(n+1)!}{n^n}$.

9. A solid cone of uniform density ρ, height H and base radius R rotates at constant angular velocity ω about its axis. Find the kinetic energy of the cone. Note: The kinetic energy of a particle of mass m moving at speed v is given by $E = \frac{1}{2}mv^2$.

10. Given a position x, y, the elevation in a portion of a canyon floor is given by

$$f(x, y) = (1 - x)^2 + 2(y - x^2)^2$$

We drive a vertical post into the ground at $P_1 = (0, 0)$, $P_2 = (1, 0)$ and at $P_3 = (1, 1)$. The portion of each post not buried in the ground is length 1. A (not necessarily level) platform is attached to the top of these three posts. If we put a fourth post at P_2 which is normal to the canyon floor, what is the exact angle between this fourth post and the upward normal of the platform?