Number: __________________________

1. Find the value of $a$ for which the integral

$$\int_{1}^{\infty} \frac{a}{x(2x + a)} \, dx$$

converges to the value of 1.

2. a. Show, by an example, that linear dependence of the columns of a matrix does not imply the linear dependence of the rows. (Note: you need to briefly indicate why the columns of your matrix are linearly dependent and the rows are not. A proof is not needed.)

b. State the rank-nullity theorem, a.k.a. the dimension theorem, for matrices.

c. Show that, if an $n \times n$ matrix $A$ is such that $A^2 = A$, then $\text{rank}(A) + \text{rank}(I - A) = n$.

3. Find the real-valued function $y : \mathbb{R} \to \mathbb{R}$ which satisfies

$$2y'' - y' - 6y = 0,$$

$$y(0) = 1,$$

$$y'(0) = 0.$$

4. Let $M$ denote the $n \times n$ matrix

$$\begin{bmatrix}
1 & 1 & \ldots & 1 \\
\vdots & \vdots & & \vdots \\
1 & 1 & \ldots & 1
\end{bmatrix}.$$

Note that $M^2 = nM$.

a. What are the possible eigenvalues of $aI_n + bM$? Here $a$ and $b$ are scalars and $I_n$ is the identity $n \times n$ matrix.

b. Let $A$ be the $4 \times 4$ matrix

$$A = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}.$$

Let $k$ be an integer greater than 1. Find $A^k$. Note: You may express your answer in the form of the product $B^xHC^y$, where $B$, $H$, $C$ are nontrivial matrices whose elements must be specified.
5. Let \( f_n(x) \) be defined recursively as \( f_n(x) = 1 - \int_0^x s[f_{n-1}(s)]ds \) with \( f_0(x) = 1 \).
   
a. Determine \( f_1(x), f_2(x), f_3(x), f_4(x) \).
   
b. What is the coefficient of the highest order term in \( f_n \)?
   
c. Show that for each \( x \in \mathbb{R} \), \( \lim_{n \to \infty} f_n(x) \) exists.

6. A curve is given by the parametric equations
   
   \[
   x = a \cos^3 t, \quad y = a \sin^3 t, \quad 0 < t < \frac{\pi}{2},
   \]

   a. Show that, near \( t = 0 \), the equations can be approximated by
   
   \[
   a - x \approx 3at^2/2, \quad y \approx at^3.
   \]

   b. Find the length of the curve. Note: A useful identity is \( \sin 2\theta = 2 \sin \theta \cos \theta \).

7. A company is looking to invest in two assets. If \( w_1 \) and \( w_2 \) are the amounts of the investment budget invested in each asset, then the variance of the total investments is given by
   
   \[
   \frac{1}{4} w_1^2 + \frac{1}{9} w_2^2 + \frac{1}{3} \rho w_1 w_2
   \]

   where \( \rho \in [-1, 1] \) is a known fixed number. The expected return of this investment is given by the expression
   
   \[
   8w_1 + 4w_2
   \]

   What is the best choice of \( w_1 \) and \( w_2 \)—in terms of \( \rho \)—to minimize the variance while ensuring an expected return of 24?

8. a. Prove that \( \sum_{n=1}^{\infty} a_n \) converges absolutely if \( \lim_{n \to \infty} n^\alpha a_n = A \) for some \( \alpha > 1 \), where \( A \) is some real number.

   b. For each of the following, determine whether the series converges or diverges and provide the justification of your answer.
      
      \( \text{(ii) } \sum_{n=1}^{\infty} \sin \left( \frac{\pi}{n^2} \right) \)
      \( \text{(ii) } \sum_{n=1}^{\infty} \frac{(n+1)!}{n^n} \).

9. A solid cone of uniform density \( \rho \), height \( H \) and base radius \( R \) rotates at constant angular velocity \( \omega \) about its axis. Find the kinetic energy of the cone. Note: The kinetic energy of a particle of mass \( m \) moving at speed \( v \) is given by \( E = \frac{1}{2}mv^2 \).
10. Given a position $x, y$, the elevation in a portion of a canyon floor is given by

$$f(x, y) = (1 - x)^2 + 2(y - x^2)^2$$

We drive a vertical post into the ground at $P_1 = (0, 0)$, $P_2 = (1, 0)$ and at $P_3 = (1, 1)$. The portion of each post not buried in the ground is length 1. A (not necessarily level) platform is attached to the top of these three posts. If we put a fourth post at $P_2$ which is normal to the canyon floor, what is the exact angle between this fourth post and the upward normal of the platform?