Qualifying Exam – Spring 2021

March 29, 2021

- The allocated time for this exam is 240 minutes.
- Write your identifying number on every page you use.
- All problems have equal weight.
- **Calculators are permitted. Show your work for full credit.**

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1. Let A and B be 3×3 real matrices that commute: $AB = BA$. If λ is a (real) eigenvalue of A, let V_λ be the real subspace of all eigenvectors having this eigenvalue.

   (a) Show that if V_λ is **one-dimensional**, then every (nonzero) vector $v \in V_\lambda$ is also an eigenvector of B, possibly with a different eigenvalue.

   (b) Give an example showing that if $\dim V_\lambda > 1$, then some vectors in V_λ may not be eigenvectors of B.

   (c) If all the eigenvalues of A are real and distinct (so each has algebraic multiplicity one), show that there is a basis in which both A and B are diagonal.

1-Solution: (a) Suppose $v \in V_\lambda \setminus \{0\}$. Since V_λ is one-dimensional, we have $V_\lambda = \text{span}\{v\}$. Note that $ABv = BAv$, and so $ABv = \lambda Bv$. Therefore, $Bv \in V_\lambda$ as well. Since v spans V_λ, hence we may write $Bv = \mu v$ for some $\mu \in \mathbb{R}$. But this means that v is also an eigenvector of B.

(b) Let A be the 3×3 identity matrix and B the diagonal matrix with the entries $1, 2$ and 3 on the diagonal. Clearly $AB = BA$. The vector $v = (1, 1, 1)^T$ is an eigenvector of A, but not of B.

(c) Since all eigenvalues are distinct, there are exactly three eigenvalues and so each eigenspace has dimension 1 and there exists an eigenbasis of A. Since by (a), each eigenvector of A is also an eigenvectors of B, this eigenbasis for A is also one for B. This basis diagonalizes both A and B.

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2. Consider the function $f(x) = x^x$.

   (a) Find $\lim_{x \to 0^+} x^x$.

   (b) Find $f'(x)$.

2-Solution: (a) (L'Hospital)

$$\lim_{x \to 0^+} x^x = \exp[\lim_{x \to 0^+} x \ln(x)] = \exp[\lim_{x \to 0^+} \frac{\ln(x)}{1/x} = \exp[\lim_{x \to 0^+} -x]] = 1.$$

(b) Write $y = x^x$, then $\ln(y) = x \ln(x)$. Implicit differentiation gives

$$(1/y)dy/dx = \ln(x) + 1,$$

so $y' = y(\ln(x) + 1) = x^x(\ln(x) + 1)$.

3. An ant walks along the surface $z = x^2 - y^2 + 4y + 14$ so that the distance from the ant to the z–axis remains constant at 2 units.

   (a) What is the highest and lowest elevation obtained by the ant during its walk?

   (b) Suppose the ant modulates its speed so that it circles the z–axis at a constant rate of $\frac{2\pi}{3}$ radians per hour. What is the longest continuous interval of time during which the ant's elevation is increasing?

3-Solution: Lagrange multipliers give:

$$2x = \lambda(2x) \text{ and } -2y + 4 = \lambda(2y).$$

If $x = 0$, get point $(0, \pm 2)$. If not, get $\lambda = 1$ and points $(\pm\sqrt{3}, 1)$. Evaluating, $f(0, -2) = 2$ is minimum value and $f(\pm\sqrt{3}, 1) = 20$ is maximum value.

For (b): The angle between the above points on the circle is $2\pi/3$, which is the span covered by the ant in one hour.

4. Find out whether the following series **converge absolutely**, **converge conditionally**, or **diverge**. In each case, explain your reasoning and give the name of the test or method you used.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n^2}\sqrt{n}}{e^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2-\cos(1/n)}{n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$

4-Solution: (a) Series **diverges** by the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{2^{(n+1)^2}\sqrt{n+1}}{e^{n+1}} \frac{e^n}{2^{n^2}\sqrt{n}} = \frac{\sqrt{1+\frac{1}{n}}}{e} 2^{2n+1} \to \infty \, (n \to \infty)$$

(b) Series **converges conditionally**:

- Series converges by alternating series test:

$$\frac{2-\cos(1/n)}{n} \to 0 \, (n \to \infty)$$

  as $2-\cos(1/n)$ is bounded and $1/n \to 0$.
  Furthermore, for $n \geq 1$, we have $\cos(1/n) < \cos(1/(n+1))$, and so $2 - \cos(1/(n+1)) < 2 - \cos(1/n)$, and thus

$$\frac{2-\cos(1/(n+1))}{n+1} < \frac{2-\cos(1/n)}{n}.$$

  So the absolute value of the underlying alternating sequence is decreasing.
- Series doesn't converge absolutely:

$$\left|\frac{2-\cos(1/n)}{n}\right| > \frac{1}{n}$$

  So by the comparison test, the series of absolute values diverges since the harmonic series $\sum 1/n$ diverges.

(c) The series $\sum_{n=1}^{\infty} e^{-2n} = \sum_{n=0}^{\infty}(1/e^2)^n$ **converges absolutely** as it is a geometric series with base $\rho = 1/e^2 < 1$.

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5. Consider the following pharmacokinetics model: A drug is ingested and enters the gastrointestinal tract (GI). From there, it is absorbed into the blood stream (B). The drug is distributed in the body and slowly removed from the blood stream by metabolization. We assume that absorption of the drug from the GI to blood happens at at rate proportional to the amount in the GI with proportionality constant $\alpha = 0.70\,\text{hour}^{-1}$. Metabolizing of the drug from the blood happens at a rate proportional to the amount in blood with proportionality constant $\beta = 0.11\,\text{hour}^{-1}$.

   Drug is ingested at the constant rate of 40 mg/hour for 15 minutes (0.25 hours) starting at $t = 0$. Let $g(t)$ denote the amount of drug in the GI (in mg) after t hours and $b(t)$ the amount of drug in the blood.

   (a) Denote by $I(t)$ the drug intake:

   $$I(t) = \begin{cases} 40\,\text{mg/hour}, & \text{for } 0 \le t \le 0.25 \\ 0 & \text{for } t > 0.25 \end{cases}$$

   The equation for $g(t)$ is then $\frac{dg}{dt} = I(t) - \alpha g$. Set up the equation for $b(t)$.

   (b) Assume the initial amount of the drug in both GI and blood is zero. What is amount of drug in the blood after 15 minutes?

   (c) What is the amount of drug in the GI after 1 hour?

5-Solution: (a) $\frac{db}{dt} = \alpha g - \beta b$

(b) For $t \le 0.25$, we have $g' = 40 - \alpha g$ with $g(0) = 0$. This gives

$$g(t) = \frac{40}{\alpha}(1 - e^{-\alpha t}).$$

Plugging this into the differential equation for $b(t)$ gives

$$\frac{db}{dt} = 40(1 - e^{-\alpha t}) - \beta b \quad \text{with } b(0) = 0.$$

The integrating factor for this equation is $\mu(t) = e^{\beta t}$, and we get

$$b(t) = \frac{40}{\beta} + \frac{40}{\alpha - \beta} e^{-\alpha t} - 40\left(\frac{1}{\beta} + \frac{1}{\alpha - \beta}\right) e^{-\beta t}.$$

Using this explicit form , we get

$$b(0.25) \approx 0.8185 \text{ mg}$$

(c) For $t > 0.25$, the equation for $g(t)$ is

$$\frac{dg}{dt} = -\alpha g \quad \text{with } g(0.25) = \frac{40}{\alpha}\left(1 - e^{-0.25\,\alpha}\right).$$

We thus have $g(t) = Ce^{-\alpha t}$ where C can be determined via the initial condition. Thus we obtain

$$g(t) = \frac{40}{\alpha}\left(e^{0.25\alpha} - 1\right)e^{-\alpha t} \quad \text{for } t \geq 0.25$$

This gives after one hour

$$g(1) \approx 5.426 \text{ mg}.$$

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6. Consider the sphere $x^2 + y^2 + (z-3)^2 = 1$. Find the point $P = (a, b, c)$ on the sphere such that the tangent plane to the sphere at P intersects the xy–plane in the line $y = 3x$.

6-Solution: Write $F(x, y, z) = x^2 + y^2 = (z-3)^2 - 1$, so the sphere is the set $\{F(x, y, z) = 0\}$. The gradient vector $\nabla F(x, y, z) = \langle 2x, 2y, 2z - 6\rangle$ is normal to this surface, so the tangent plane at $P = (a, b, c)$ is given by

$$2a(x-a) + 2b(y-b) + (2c-6)(z-c) = 0.$$

Setting $z = 0$ gives the line

$$ax + by = a^2 + b^2 + c^2 - 3c = 1 - (c-3)^2 + c^2 - 3c = 3c - 8.$$

in the xy-plane (by the equation of the sphere).

It follows that $c = 8/3$. We also need to have $-a/b = 3$, so $a = -3b$. Plugging this into the equation of the sphere and solving:

$$a^2 + b^2 = 8/9 \to b = \sqrt{8/90} \to a = \sqrt{72/90}.$$

So, $P = \left(\dfrac{2\sqrt{5}}{5}, \dfrac{2\sqrt{5}}{15}, \dfrac{8}{3}\right)$.

7. Let V be a vector space and $\ell\colon V \to \mathbb{R}$ be a linear map. If $z \in V$ is not in the nullspace of ℓ, show that every $x \in V$ can be decomposed *uniquely* as $x = v + cz$, where $v \in V$ is in the nullspace of ℓ and $c \in \mathbb{R}$ is a scalar.

7-Solution: Note that $\ell(z) \neq 0$. Let $x \in V$.

- **Existence of the decomposition:** We claim that

$$x = \left(x - \frac{\ell(x)}{\ell(z)} z\right) + \frac{\ell(x)}{\ell(z)} z$$

is the desired decomposition. Indeed, note that $\ell\left(x - \frac{\ell(x)}{\ell(z)} z\right) = 0$ by linearity and so $x - \frac{\ell(x)}{\ell(z)} z$ is in the nullspace of ℓ.

- **Uniqueness:** Suppose $x = v_1 + c_1 z = v_2 + c_2 z$, where v_1 and v_2 are in the nullspace of ℓ. Then $\ell(x) = c_1 \ell(z) = c_2 \ell(z)$. This means $c_1 = c_2$. This in turn implies $v_1 = x - c_1 z = x - c_2 z = v_2$.

8. Consider the three functions $a(x)$, $b(x)$, and $c(x)$ given by:

$$a(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} \qquad b(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \qquad c(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

(a) Determine the radius of convergence for each of function.

(b) Give a formula for the n^{th} derivative $a^{(n)}(x)$ for all $n \in \mathbb{N}$.

(c) Show that the following identity holds for all x in the radius of convergence:

$$a^3 + b^3 + c^3 - 3abc = 1$$

8-Solution: (a) By ratio test:

$$\lim_{n\to\infty} \left| \frac{(3n+\varepsilon)!}{x^{3n+\varepsilon}} \frac{x^{3n+3+\varepsilon}}{(3n+3+\varepsilon)!} \right| = \lim_{n\to\infty} \frac{|x|^3}{(3n+3+\varepsilon)(3n+2+\varepsilon)(3n+1+\varepsilon)} = 0,$$

so the interval of convergence is infinite for $\varepsilon = 0, 1, 2$.

(b)

$$a^{(n)}(x) = \begin{cases} a(x) & \text{if } n \cong 0 \pmod 3 \\ c(x) & \text{if } n \cong 1 \pmod 3 \\ b(x) & \text{if } n \cong 2 \pmod 3 \end{cases}$$

(c) Use implicit differentiation on the desired formula:

$$3a^2a' + 3b^2b' + 3c^2c' - 3(a'bc + ab'c + abc') = 3a^2c + 3b^2a + 3c^2b - 3c^2b - 3a^2c - 3b^2a = 0$$

using the fact that $a' = c$, $b' = a$, and $c' = b$. By the fundamental theorem of calculus, we have

$$a^3 + b^3 + c^3 - 3abc = C$$

for some constant C, but evaluating at $x = 0$ indicates that $C = 1$.

9. Let

$$Z_1 = \{(x,y,z) \in \mathbb{R}^3 \text{ such that } x^2 + y^2 \leq z\}$$

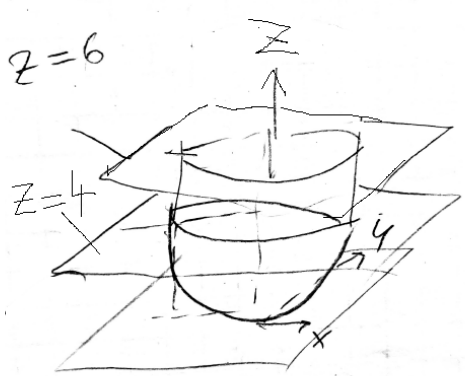
and

$$Z_2 = \{(x,y,z) \in \mathbb{R}^3 \text{ such that } -6 \leq z \leq 6, x^2 + y^2 \leq 4\}$$

(a) Sketch the intersection $B = Z_1 \cap Z_2$.

(b) Find the **z–coordinate** $\bar{z}$ of the center of mass of B.

9-Solution: (a) The set B is the union of a paraboloid $z \leq x^2 + y^2$ cut off at the plane $x = 4$ and a cylinder $x^2 + y^2 \leq 4$ with $4 \leq z \leq 6$.



(b) The volume of B is $Vol(B) = V_1 + V_2$ with

$$V_1 = \int_0^4 \int_0^{2\pi} \int_0^{\sqrt{z}} r dr\, d\theta\, dz = 8\pi \quad \text{(paraboloid)}$$

and

$$V_2 = \int_4^6 \int_0^{2\pi} \int_0^{2} r dr\, d\theta\, dz = 8\pi \quad \text{(cylinder)}$$

This gives the $z-$coordinate of the center of mass as

$$\bar{z} = \frac{1}{Vol(B)} \int_B z\, dV =$$

$$= \frac{1}{16\pi} \left(\int_0^4 \int_0^{2\pi} \int_0^{\sqrt{z}} zr dr\, d\theta\, dz + \int_4^6 \int_0^{2\pi} \int_0^{2} zr dr\, d\theta\, dz \right) = \frac{23}{6}.$$

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10. For $n \geq 8$, determine which is larger: $(\sqrt{n+1})^{\sqrt{n}}$ or $(\sqrt{n})^{\sqrt{n+1}}$. Justify your answer.

10-Solution: We claim $(\sqrt{n+1})^{\sqrt{n}} < (\sqrt{n})^{\sqrt{n+1}}$. This is equivalent to

$$\frac{\ln(n)}{\sqrt{n}} > \frac{\ln(n+1)}{\sqrt{n+1}}.$$

So, we consider $f(x) = x^{-1/2}\ln(x)$. This function is decreasing for $x \geq 8$ (product rule).