WWU Math Graduate Qualifying Exam, Fall 2020

Note: All questions have the same value (10 pts). One completely solved (or mostly completely solved) solution is viewed as more valuable that two partially solved problems, even if the number of points earned for the two problems exceeds what's earned for the complete solution.

Instructions: repeated here are the instructions you received by email:

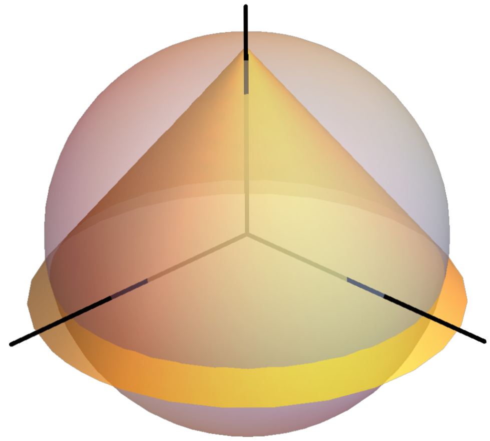
- You may use calculators, but you may not use high-end mathematical software like Mathematica, or Gage, or Maple etc. Note: if you wish to use software to simply graph a function, that is OK, but we trust you to limit use to the equivalent of what you could do on a graphing calculator.
- It is expected that you demonstrate how you got your answer an answer without any work will not be accepted, within reason of course. Please use plain, white, unlined paper for your answers. Make sure your answers are clearly numbered. As usual, if you end up having more than one attempt at a problem, cross out the version you dont want us to grade. It is not acceptable to submit two different answers.
- If a question is unclear, you may email either Steve McDowall (mcdowas@wwu.edu) or Yuan Pei (peiy@wwu.edu) to ask for clarification. We will do our best to get back to you as soon as we can. When you are finished, use your phone, or a scanner, to scan your pages.
- You will not be proctored during the exam. You have agreed that you will work on your own, without use of unauthorized technology, without searching on the internet for help, or acting in any other way understood to be outside the expectations for an examination.

A reminder of submission instructions: when you have completed the exam you will need to scan it and email the single PDF file to Rick Barnard (barnarr3@wwu.edu).

- Important: we do not want to receive multiple jpeg photos of your pages; we want to receive a single PDF. I suggest you use the (free) app "CamScanner." With this app you can specify that you are taking multiple images (which makes a single document). You can then email the document as a PDF.
- You can make your email submission Between 12:00 pm and 1:00 pm (to barnarr3@wwu.edu). This preserves your anonymity as Rick is not involved in grading the exam. Be sure to include the code you registered with Melissa.
- When you have submitted your exam to Rick, wait for a response from him verifying that your exam has been received and that it is high enough quality to enable us to grade it. If it is not, Rick will let you know and ask you to try scanning it again perhaps consider the lighting for example.

Exam questions begin on the following page.

1. Consider an "upside down" cone whose point is at the "North Pole" ($(0,0,1)$) on the unit sphere, and which intersects the sphere again at $z=-a$ ($0<a<1$). See the picture. This cone divides the sphere into two regions: that *above* the cone and that *below* the cone. Find the value of a so that these two regions have the same volume.



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2. (a) Use calculus to verify that $\frac{2}{\pi}\theta \le \sin\theta \le \theta$ when $0 \le \theta \le \frac{\pi}{2}$.

   (b) For $\lambda < 1$, use (a) to prove that

$$\lim_{T\to\infty} T^{\lambda} \int_0^{\frac{\pi}{2}} e^{-T\sin\theta}\, d\theta = 0.$$

   (You may use without further justification facts such as $e^{-x} \to 0$ as $x \to \infty$.)

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3. (a) Explain why the equation $x^n + x^{n-1} + \cdots + x = 1$ has only one real root in the interval $(0.5, 1)$ for positive integer $n > 1$.

   (b) Denote by x_n the above root, determine the value of $\lim_{n\to\infty} x_n$ and explain your answer.

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4. Given an equilateral triangle T_0 with each side of length L, remove the middle one-third section of each side and attach a smaller equilateral triangle of side-length $L/3$ and obtain a star-shaped symmetric hexagon, denoted by T_1; repeat the above process to each of the six small triangles at each vertex, adding new triangles to the outside edges, and get T_2, as shown in the picture. Continue this process to obtain $T_n \ldots$.

   (a) Denote by S_n the circumference of T_n and determine whether $\lim_{n\to\infty} S_n$ is finite or not. If finite, find its value. Justify your conclusion.

   (b) Denote by A_n the area of T_n and determine whether $\lim_{n\to\infty} A_n$ is finite or not. If finite, find its value. Justify your conclusion.

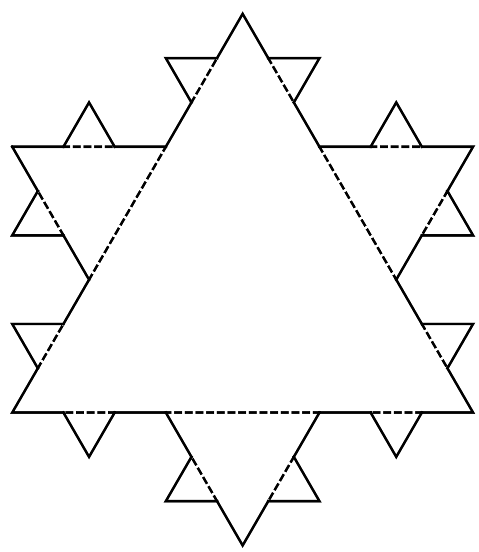

Figure: T_2

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5. (a) Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{\ln(1+n)}.$$

(Hint, compare with another series; the integral test may be helpful.)

(b) Let

$$s_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx \quad \text{with} \quad n \in \{1, 2, 3, \ldots\}.$$

Evaluate the series $\displaystyle\sum_{n=1}^{\infty} \frac{s_n + s_{n+2}}{n}$. (Hint, $1 + \tan^2 x = \sec^2 x$.)

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6. An $n \times n$ symmetric matrix $\mathbf{P}$ for which $\mathbf{P}^2 = \mathbf{P}$ is called a *projection matrix.*

(a) Show that if $\mathbf{P}$ is a projection matrix then all its eigenvalues are either 0 or 1.

(b) Let $\mathbf{P}$ be an $n \times n$ projection matrix which has rank r. Show that exactly r of $\mathbf{P}$'s eigenvalues are 1 and exactly $n - r$ are 0.

(c) Let $\vec{u} \in \mathbb{R}^n$ be a unit vector. Define $A = \vec{u}\vec{u}^T$.

i. Determine whether or not A is necessarily a projection matrix. Justify your answer.

ii. Find an eigenvalue of A and a corresponding eigenvector, justifying your claim.

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7. For a linear transformation $\mathbf{A}$ on a vector space V, a subspace W of V is called $\mathbf{A}$-invariant if $\mathbf{A}W \subset W$, i.e., for any vector $\vec{w} \in W$, $\mathbf{A}\vec{w} \in W$.

(a) Consider a linear transformation on $\mathbb{R}^4$ with standard matrix $\mathbf{A}$ under the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{bmatrix}.$$

Verify that the subpace $W = \text{span}\{\vec{v}_1 + 2\vec{v}_2, \vec{v}_2 + \vec{v}_3 + 2\vec{v}_4\}$ is $\mathbf{A}$-invariant.

(b) Now consider a linear transformation $\mathbf{K}$ on $\mathbb{R}^n$ under the basis $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n\}$,

$$\mathbf{K} = \begin{bmatrix} k & 1 & 0 & \cdots & 0 & 0 \\ 0 & k & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k & 1 \\ 0 & 0 & 0 & \cdots & 0 & k \end{bmatrix},$$

where k is a scalar. Show that:

i. if some $\mathbf{K}$-invariant subspace W contains $\vec{u}_n$, then $W = \mathbb{R}^n$;

ii. $\vec{u}_1$ belongs to *any* non-trivial $\mathbf{K}$-invariant subspace of $\mathbb{R}^n$.

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8. Let E be the ellipsoid given by the equation $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1$. Let B be a rectangular box, centered at the origin with sides parallel to the coordinate axes, of dimensions $l \times w \times h$. Find the dimensions of the box of maximal volume which fits within the ellipsoid E.

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9. Let E be the ellipsoid $x^2 + 16y^2 + z^2 = 16$, and P be the plane $x - 2y - z = 3$, which does not intersect E. Find the point on E which is closest to P, and the (perpendicular) distance from p to P. (Hint: at p the tangent plane to E must be parallel to P.)

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10. Consider a fox chasing a rabbit; the fox's path is $F(t) = (x(t), y(t))$ with $F(0) = (0, 0)$, and the rabbit's path is $(1, t)$. The fox always runs *directly toward* the rabbit at a speed that is twice the distance from the fox to the rabbit; that is, at all time t, the velocity vector of the fox is directly toward the position of the rabbit at that time and its length is twice the distance from the fox to the rabbit. Use this information to find the path of the fox, $F(t)$. Does the fox ever catch the rabbit? If so, when? If not, explain.