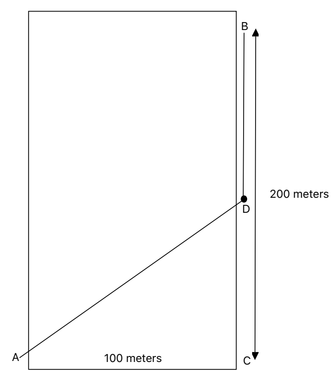
This exam starts at 9 a.m. and ends at 1 p.m. Calculators may not be used. Please provide complete answers that show your work. Clearly identify each problem you are working on on your page. Write your secret ID on **every page of your exam**.

**Problem 1.** (10 points) A pipe will cross from point A to point B on the two sides of a river (see the figure), passing through the point D (which is to be found). The river has parallel banks and its (constant) width is 100m. The point C is directly across the river from the point A and the distance from C to B is 200m. The pipe costs \$1000 per meter on land (between D and B) and \$1500 per meter over the river (between A and D).

Where should D be placed to minimize the cost of the pipe (in this portion between A and B)? Your work should clearly define variables and describe the steps taken as well as reporting the location of D.




*Solution:* Let $x$ denote the distance from C to D. The length of the pipe over the river is $\sqrt{100^2 + x^2}$ and the length along over land is $200 - x$. The combined cost is therefore

$$f(x) = 1500\sqrt{100^2 + x^2} + 1000\,(200 - x)\,.$$

Critical points for $f$ satisfy

$$0 = f'(x) = \frac{1500\,x}{\sqrt{100^2 + x^2}} - 1000\,, \text{ or } 1500\,x = 1000\,\sqrt{100^2 + x^2}\,.$$

For $x > 0$, this equation is equivalent to $1.5^2\,x^2 = 100^2 + x^2$ and $x^2 = 100^2/1.25$ so $x = 200/\sqrt{5}$.

For $x < 0$, $f'(x) < 0$, and $f'(200) = 1000((3/\sqrt{5}) - 1) > 0$, the minimum occurs for $x > 0$ and the critial point is a minimum. We conclude that D should be placed a distance of $200/\sqrt{5}$ meters from C, and toward B. □

**Problem 2.** This problem has three parts.
(a) (4 points) Clearly state the mean value theorem, relating the mean value and derivative of a function.
(b) (3 points) Does there exist a differentiable function $f : \mathbf{R} \to \mathbf{R}$ with $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all $x$? Prove your answer.
(c) (3 points) Does there exist a differentiable function $f : \mathbf{R} \to \mathbf{R}$ with $f(0) = -1$, $f(2) = 4$ and $f'(x) \geq 2$ for all $x$? Prove your answer.

*Solution.* (a) Omitted. (b) No, there does not. By the mean value theorem there must be a point $c \in (0, 2)$ with $f'(c) = (4 - (-1))/(2 - 0) = 5/2 > 2$. (c) Yes, there does. Take the linear function $f(x) = 5x/2 - 1$. □

**Problem 3.** For $t \geq 1$, consider the function

$$f(t) = \int_1^2 \frac{1+5t}{1+t\,x^2}\,dx + \int_1^t \frac{3}{1+x^3}\,dx.$$

(a) (6 points) Show that $f(t)$ is strictly increasing over its domain.
(b) (4 points) Is $f(t)$ bounded for $t \geq 1$? Explain.

*Solution.* Use differentiating under the integral sign and the fundamental theorem of calculus. We get

$$f'(t) = \int_1^2 \frac{5-x^2}{(1+t\,x^2)^2}\,dx + \frac{3}{1+t^3} > 0,$$

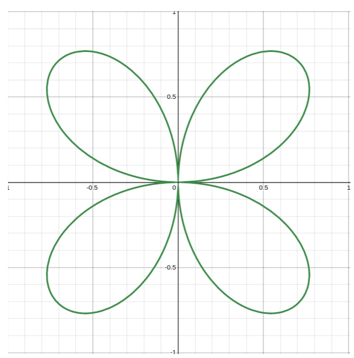
where we used that $5 - x^2 > 0$ over $[1, 2]$, and that $t \geq 1$.

Therefore, (a) $f$ is increasing for $t \geq 1$. Also, $(1 + 5t)/(1 + tx^2)$ reaches its maximum in $x$ over $[1, 2]$ at $x = 1$ (for any $t$ with $t > 0$). Thus

$$f(t) = \int_1^2 \frac{1+5t}{1+t\,x^2}\,dx + \int_1^t \frac{3}{1+x^3}\,dx \leq \int_1^2 \frac{1+5t}{1+t}\,dx + \int_1^\infty \frac{3}{1+x^3}\,dx.$$

Since the last two integrals are bounded, so is $f(t)$ for $t \geq 1$. □

**Problem 4.** This problem deals with the (four petaled) rose, which is the curve shown below.



This curve has parameterization

$$x = \sin(2t)\cos(t), \quad y = \sin(2t)\sin(t), \qquad t \in \mathbf{R}.$$

(a) (3 points) Show that $(x(t)^2 + y(t)^2)^3 = 4x(t)^2 y(t)^2$. (You need not prove this, but any point $(x, y)$ satisfying $(x^2 + y^2)^3 = 4x^2y^2$ must lay on the rose.)
(b) (4 points) Compute the tangent line to the curve at the point $(1/\sqrt{2}, 1/\sqrt{2})$. No eye-balling it.
(c) (3 points) Set up, but do not compute or simplify, an integeral that computes the arc length of one of the petals. Your integral should be in terms of the usual trigonometric functions and arithmetic operations.

*Solution.* (a) We plug into the equation and get

$$\begin{aligned}
(\sin^2(2t)\cos^2(t) + \sin^2(2t)\sin^2(t))^3 &= \sin^6(2t)(\cos^2(t) + \sin^2(t))^3 \\
&= \sin^6(2t) \\
&= (\sin^3(2t))^2 \\
&= (\sin^2(2t)\sin(2t)) \\
&= (\sin^2(2t) \cdot 2\sin(t)\cos(t)) \\
&= (2 \cdot \sin(2t)\cos(t) \cdot \sin(2t)\sin(t))^2 = (2x(t)y(t))^2.
\end{aligned}$$

(b) The slope of the tangent line can be found by implicitly differentiating the equation $(x^2+y^2)^3 - 4x^2y^2 = 0$ or using the formula $dy/dx = y'(t)/x'(t)$ and setting $t = \pi/4$. The latter formula gives,

$$(dy/dt)/(dx/dt) = \frac{2\cos(2t)\sin(t) + \sin(2t)\cos(t)}{2\cos(2t)\cos(t) - \sin(2t)\sin(t)}$$

and setting $t = \pi/4$ gives

$$dy/dx = \frac{0 + 1/\sqrt{2}}{-1/\sqrt{2}} = -1$$

Then tangent line is thus $y = -1(x - 1/\sqrt{2}) + 1/\sqrt{2} = -x + \sqrt{2}$.

(c) One petal of the rose is traversed by $0 \le t \le \pi$. The arc-length formula gives $\int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$. Use the above formulas for $x'(t)$ and $y'(t)$. □

**Problem 5.** Consider the following three sums.
(a) (3 points) Evaluate: $\sum_{n=0}^{\infty} \frac{1}{3^n}$.
(b) (3 points) Evaluate: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{6}\right)^{2n+1}$.
(c) (4 points) Does $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ converge? (Hint: $\frac{d}{dx}\ln(x) = ?$)

*Answers.* $3/2, 1/2$ and it diverges by the integeral test (use a $u$-substitution with $u = \ln(x)$). □

**Problem 6.** The matrix $A$ is $2 \times 3$, and its image contains all of $\mathbf{R}^2$ (the transformation is onto, or surjective). You are given that

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Describe all solutions, $\mathbf{x}$, to $A\mathbf{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. Explain how you know that you have found all the solutions.

*Solution.* Since $A$ is onto, the only solutions to the homogeneous equation are multiples of the given solution to it. By linearity, all solutions are

$$\mathbf{x} = 2\,[1,0,1] + s\,[1,2,3] = [2+s, 2s, 2+3s] \quad \text{with arbitrary } s.$$

It is also possible to find a matrix that satisfies the properties given for $A$ and then solve for $\mathbf{x}$, which gives a correct solution, though in a very messy way. □

**Problem 7.** Ler $R$ be the region $R = \{(x,y) \mid x \ge 0,\ y \ge 0,\ x + 2y \le 4\}$. Set $f(x,y) = x^2 - 2x - y + y^2$ and set $g(x,y) = xy$.
(a) (5 points) Find the maximum value of $g$ on $R$ and the point(s) at which this maximum occurs.
(b) (5 points) Find the minimum value of $f$ on $R$ and the point(s) at which this value occurs.

*Solution.* For both functions we consider critical points in the interior of $R$, critical points on the line segments in the boundary of $R$ using Lagrange multipliers, and the values at the corners of $R$, namely the points $(0,0)$, $(4,0)$, and $(0,2)$.

(a) The level sets of $g$ show that the maximum occurs on $x + 2y = 4$. So we have $y = \lambda$, $x = 2\lambda$, and $x + 2y = 4$. We conclude that $x = 2y$ and $y = 1$, and the maximum is $g(2,1) = 2$.

(b) The interior critical point for $f$ satisfies $2x - 2 = 0$ and $-1 + 2y = 0$ so it is $(1, 1/2)$. We calculate $f(1,1/2) = -5/4$. (Also, $f(0,0) = 0$, $f(4,0) = 8$, and $f(0,2) = 2$.) On $x = 0$ the minimum value is $f(0,1/2) = -1/4$. On $y = 0$ the minimum value is $f(1,0) = -1$. Finally, on $x + 2y = 4$, $2x - 2 = \lambda$, $2y - 1 = 2\lambda$, and $x + 2y = 4$. We get $2y - 1 = 4x - 4$, $y = 2x - 1.5$, and $x + 4x - 3 = 4$. So the critical point is $(1.4, 1.3)$ where $f(1.4, 1.3) = -0.45$. Therefore, the minimum value of $f$ is $f(1,1/2) = -5/4$. □

**Problem 8.** (a) (5 points) Evaluate the integral of $f(x,y) = e^{x^2}$ over the region $T = \{(x,y) \,|\, 0 \le y \le 1,\ y \le x \le 1\}$.
(b) (5 points) Evaluate the integral of $g(x,y) = x$ over the region $D = \{(x,y) \,|\, x^2 + y^2 \le 4,\ x \ge 1\}$.

Solution:

(a) $$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^x e^{x^2}\, dy\, dx = \int_0^1 x\, e^{x^2}\, dx = 0.5\, e^{x^2}\Big]_0^1 = 0.5(e-1).$$

(b) In polar coordinates, $D$ has angle $-\pi/3 \le \theta \le \pi/3$. For the radius, the line $x = 1$ becomes $r\cos(\theta) = 1$. The integral is

$$\int_{-\pi/3}^{\pi/3} \int_{1/\cos(\theta)}^{2} r\cos(\theta)\, r\, dr\, d\theta = \int_{-\pi/3}^{\pi/3} \frac{1}{3} r^3 \Big]_{1/\cos(\theta)}^{2} \cos(\theta)\, d\theta$$
$$= \int_{-\pi/3}^{\pi/3} \left( \frac{8}{3}\cos(\theta) - \frac{1}{3\cos^2(\theta)} \right) d\theta = \frac{8}{3}\sin(\theta) - \frac{1}{3}\tan(\theta)\Big]_{-\pi/3}^{\pi/3} = 2\sqrt{3}.$$

Alternatively, $D$ can be decomposed into the sector, $A$, with $-\pi/3 \le \theta \le \pi/3$ and $0 \le r \le 2$, from which we then remove the triangle, $B$, with vertices $(0,0)$, $(1,\sqrt{3})$, and $(1,-\sqrt{3})$. It is easiest to integrate over $A$ in polar coordinates and over $B$ in rectangular coordinates.

**Problem 9**. Let $A$ be a **real** 2-by-2 matrix satisfying $A^4 = 16 \cdot I$ (where $I$ is, of course, the identity matrix).

(a) (3 points) What are the possible (complex) eignenvalues for $A$?
(b) (4 points) For each eigenvalue $\lambda$ you listed above, give an example of a **real** 2-by-2 matrix $A$ with $\lambda$ as an eigenvalue so that matrix satisfies $A^4 = 16 \cdot I$.
(c) (3 points) Determine the **real** eigenvectors of the matrices you gave in (b), or explain why no real eigenvectors exist.

*Solution.* The possible eigenvalues are $2, -2, 2i$ and $-2i$ since $Av = \lambda v$ implies $A^4 v = \lambda^4 v = 16v$ and hence $\lambda^4 = 16$. This means $(\lambda^2 - 4)(\lambda^2 + 4) = 0$, which means $\lambda = \pm 2$ or $\lambda = \pm 2i$. Examples of matrices satisying the given constrains are

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

(c) Omitted. $\square$

**Problem 10.** Consider the differential equation $x'' + 2px' + x = 3$, where $x = x(t)$ and $p$ is a real constant satisfying $\cancel{|p| < 1.}$ $0<p<1$
(a) (2 points) Show that there is a constant solution $x(t) \equiv c$ to the above equation.
(b) (8 points) Prove that every solution $x(t)$ converges to the constant solution $\lim_{t\to\infty} x(t) = c$.

*Solution.* (a) We have $x(t) \equiv 3$ is a solution.

(b) We need to solve the equation $r^2 + 2pr + 1 = 0$, which has roots $\overset{-p}{\cancel{-1}} \pm \sqrt{p^2 - 1}$. Since $|p| < 1$, the roots will be distinct and real and the real part will be $-1$. The general solution of the equation will then be $x(t) = 3 + c_1 e^{-t}\cos((1-p^2)^{1/2} t) + c_2 e^{-t}\cos((1-p^2)^{1/2} t) \to 3$ as $t \to \infty$. $\square$