# Qualifying Exam – Spring 2021

March 29, 2021

- The allocated time for this exam is 240 minutes.
- Number your pages and make sure you scan all of them in the correct order.
- All problems have equal weight.
- **Graphing calculators are permitted. Show your work for full credit.**

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1. Let $A$ and $B$ be $3 \times 3$ real matrices that commute: $AB = BA$. If $\lambda$ is a real eigenvalue of $A$, let $V_\lambda$ be the real subspace of all eigenvectors having this eigenvalue.

   (a) Show that if $V_\lambda$ is **one-dimensional**, then every (nonzero) vector $v \in V_\lambda$ is also an eigenvector of $B$, possibly with a different eigenvalue.

   (b) Give an example showing that if $\dim V_\lambda > 1$, then some vectors in $V_\lambda$ may not be eigenvectors of $B$.

   (c) If all the eigenvalues of A are real and distinct (so each has algebraic multiplicity one), show that there is a basis in which both $A$ and $B$ are diagonal.

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2. Consider the function $f(x) = x^x$.

   (a) Find $\lim_{x \to 0^+} x^x$.

   (b) Find $f'(x)$.

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3. An ant walks along the surface $z = x^2 - y^2 + 4y + 14$ so that the distance from the ant to the $z$–axis remains constant at 2 units.

   (a) What is the highest and lowest elevation obtained by the ant during its walk?

   (b) Suppose the ant modulates its speed so that it circles the $z$–axis at a constant rate of $\frac{2\pi}{3}$ radians per hour. What is the longest continuous interval of time during which the ant's elevation is increasing?

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4. Find out whether the following series **converge absolutely**, **converge conditionally**, or **diverge**. In each case, explain your reasoning and give the name of the test or method you used.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n^2}\sqrt{n}}{e^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2-\cos(1/n)}{n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$

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5. Consider the following pharmacokinetics model: A drug is ingested and enters the gastrointestinal tract (GI). From there, it is absorbed into the blood stream (B). The drug is distributed in the body and slowly removed from the blood stream by metabolization. We assume that absorption of the drug from the GI to blood happens at at rate proportional to the amount in the GI with proportionality constant $\alpha = 0.70\,\text{hour}^{-1}$. Metabolizing of the drug from the blood happens at a rate proportional to the amount in blood with proportionality constant $\beta = 0.11\,\text{hour}^{-1}$.

   Drug is ingested at the constant rate of 40 mg/hour for 15 minutes (0.25 hours) starting at $t = 0$. Let $g(t)$ denote the amount of drug in the GI (in mg) after $t$ hours and $b(t)$ the amount of drug in the blood.

   (a) Denote by $I(t)$ the drug intake:

   $$I(t) = \begin{cases} 40\,\text{mg/hour}, & \text{for } 0 \le t \le 0.25 \\ 0 & \text{for } t > 0.25 \end{cases}$$

   The equation for $g(t)$ is then $\dfrac{dg}{dt} = I(t) - \alpha g$. Set up the equation for $b(t)$.

   (b) Assume the initial amount of the drug in both GI and blood is zero. What is amount of drug in the blood after 15 minutes?

   (c) What is the amount of drug in the GI after 1 hour?

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6. Consider the sphere $x^2 + y^2 + (z-3)^2 = 1$. Find the point $P = (a, b, c)$ on the sphere such that the tangent plane to the sphere at $P$ intersects the $xy$–plane in the line $y = 3x$.

7. Let V be a vector space and $\ell\colon V \to \mathbb{R}$ be a linear map. If $z \in V$ is **not** in the nullspace of $\ell$, show that every $x \in V$ can be decomposed **uniquely** as $x = v + cz$ , where $v \in V$ is in the nullspace of $\ell$ and $c \in \mathbb{R}$ is a scalar.

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8. Consider the three functions $a(x)$, $b(x)$, and $c(x)$ given by:

$$a(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} \qquad b(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \qquad c(x) = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}$$

   (a) Determine the radius of convergence for each of function.

   (b) Give a formula for the $n^{\text{th}}$ derivative $a^{(n)}(x)$ for all $n \in \mathbb{N}$.

   (c) Show that the following identity holds for all $x$ in the radius of convergence:

$$a^3 + b^3 + c^3 - 3abc = 1$$

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9. Let

$$Z_1 = \{(x, y, z) \in \mathbb{R}^3 \text{ such that } x^2 + y^2 \leq z\}$$

and

$$Z_2 = \{(x, y, z) \in \mathbb{R}^3 \text{ such that } -6 \leq z \leq 6, x^2 + y^2 \leq 4\}$$

   (a) Sketch the intersection $B = Z_1 \cap Z_2$.

   (b) Find the **z–coordinate** $\bar{z}$ of the center of mass of $B$.

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10. For $n \geq 8$, determine which is larger: $(\sqrt{n+1})^{\sqrt{n}}$ or $(\sqrt{n})^{\sqrt{n+1}}$. Justify your answer.