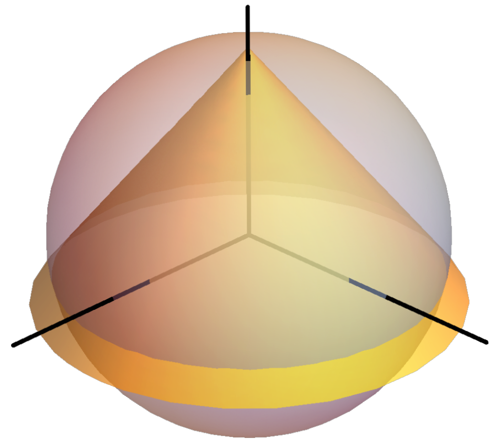
WWU Math Graduate Qualifying Exam, Fall 2020 – Solutions

1. Consider an "upside down" cone whose point is at the "North Pole" ($(0, 0, 1)$) on the unit sphere, and which intersects the sphere again at $z = -a$ ($0 < a < 1$). See the picture. This cone divides the sphere into two regions: that *above* the cone and that *below* the cone. Find the value of a so that these two regions have the same volume.



Solution

(5 pts) Set up a correct integral.

(2.5 pts) Correct evaluation of the integral.

(2.5 pts) Correct determination of a (consistent with previous work).

In cylindrical coordinates, or simply using "washers" parallel to the xy-plane, the volume in the sphere *outside* the cone is

$$\pi\int_{-a}^{1}\left[(1-z^2)-\frac{1-a^2}{(1+a)^2}(1-z)^2\right]dz = \pi\left[z-\frac{1}{3}z^3+\frac{1-a^2}{3(1+a)^2}(1-z)^3\right]_{z=-a}^{z=1}$$
$$= \pi\left[\frac{2}{3}+a-\frac{a^3}{3}-\frac{(1-a^2)(1+a)}{3}\right].$$

Setting this equal to half the volume of the sphere, $\frac{2\pi}{3}$, we obtain $a^2+2a-1=0$ so $a=-1\pm\sqrt{2}$; since $a>0$, $a=-1+\sqrt{2}$.

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2. (a) Use calculus to verify that $\frac{2}{\pi}\theta \le \sin\theta \le \theta$ when $0 \le \theta \le \frac{\pi}{2}$.

   (b) For $\lambda < 1$, use (a) to prove that

$$\lim_{T\to\infty} T^{\lambda}\int_0^{\frac{\pi}{2}} e^{-T\sin\theta}\,d\theta = 0.$$

   (You may use without further justification facts such as $e^{-x}\to 0$ as $x\to\infty$.)

Sketch of solution:

(a) (6 pts) It is true if $\theta = 0$. For $0 < \theta \le \frac{\pi}{2}$, it suffices to consider $f(\theta) = \frac{\sin\theta}{\theta}$ and show that $\frac{2}{\pi} \le f(\theta) \le 1$. Calculate $f'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2} := \frac{g(\theta)}{\theta^2}$. On $(0, \frac{\pi}{2}]$, g is strictly decreasing so $g(\theta) < 0$ so $f' < 0$.

(b) (4 pts) By (a), we know $e^{-T\sin\theta} \le e^{-T\frac{2}{\pi}\theta}$. Then, use squeeze theorem to obtain the answer 0.

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3. (a) Explain why the equation $x^n + x^{n-1} + \cdots + x = 1$ has only one real root in the interval $(0.5, 1)$ for positive integer $n > 1$.

   (b) Denote by x_n the above root, determine the value of $\lim_{n\to\infty} x_n$ and explain your answer.

Sketch of solution:

(a) (4 pts) Let $f(x) = x^n + x^{n-1} + \cdots + x - 1$, so $f(x)$ is continuous on $[0.5, 1]$ and $f(1) > 0$ while $f(0.5) < 0$; also, $f'(x) > 0$, so (a) is true.

(b) (6 pts)By $\sum_{k=1}^{n} x_n^k = 1$ and $0.5 < x_n < 1$, we know $\sum_{k=0}^{n} x_n^k = 2$, which implies $x_n = \frac{1}{2} + \frac{x_n^{n+1}}{2}$. Also, by $\sum_{k=1}^{n+1} x_{n+1}^k = 1$ and $0.5 < x_{n+1} < 1$, we know $\sum_{k=1}^{n} x_{n+1}^k < 1$, which implies $x_{n+1} < x_n$, so $\lim_{n\to\infty} x_n = \frac{1}{2}$.

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4. Given an equilateral triangle T_0 with each side of length L, remove the middle one-third section of each side and attach a smaller equilateral triangle of side-length $L/3$ and obtain a star-shaped symmetric hexagon, denoted by T_1; repeat the above process to each of the six small triangles at each vertex, adding new triangles to the outside edges, and get T_2, as shown in the picture. Continue this process to obtain $T_n \ldots$.

   (a) Denote by S_n the circumference of T_n and determine whether $\lim_{n\to\infty} S_n$ is finite or not. If finite, find its value. Justify your conclusion.

   (b) Denote by A_n the area of T_n and determine whether $\lim_{n\to\infty} A_n$ is finite or not. If finite, find its value. Justify your conclusion.

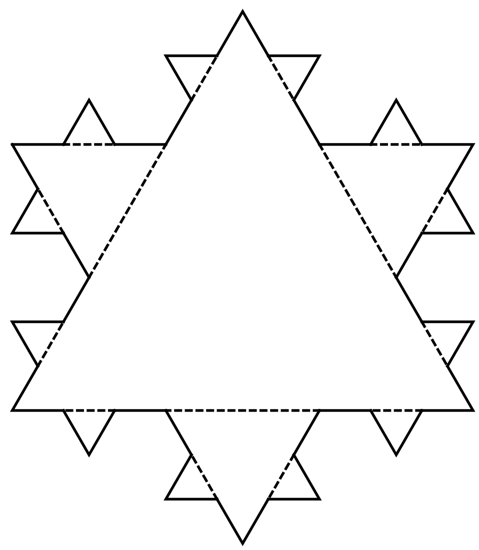

Figure: T_2

Sketch of solution:

(a) (3 pts) deriving the correct series; (1 pt) conclusion. We find $S_n = 3L\left(\frac{4}{3}\right)^n \to \infty$.

(b) (4 pts) deriving the correct series; (2 pts) computing sum: We find $A_0 = \frac{\sqrt{3}}{4}L^2$ and $A_n = A_{n-1} + \frac{3\sqrt{3}L^2}{16}\left(\frac{4}{9}\right)^n$. Thus the total area is

$$A_0 + \frac{3\sqrt{3}L^2}{16}\sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n - 1 = \frac{2\sqrt{3}}{5}L^2.$$

5. (a) Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{\ln(1+n)}.$$

(Hint, compare with another series; the integral test may be helpful.)

(b) Let

$$s_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx \quad \text{with} \quad n \in \{1, 2, 3, \dots\}.$$

Evaluate the series $\displaystyle\sum_{n=1}^{\infty} \frac{s_n + s_{n+2}}{n}$. (Hint, $1 + \tan^2 x = \sec^2 x$.)

Sketch of solution:

(a) (3 pts) choosing a test that will work; (2 pts) using the test to get the correct conclusion: e.g. Divergent; compare with $\sum\limits_{n=1}^{\infty} \dfrac{1}{(1+n)\ln(1+n)}$, and the latter is divergent due to the integral test $\int\limits_1^{\infty} \dfrac{1}{(1+x)\ln(1+x)} dx = \infty$.

(b) (3.5 pts) computing the relevant integral; (1.5 pts) evaluating the resulting series: $s_n + s_{n+2}$ can be explicitly evaluated using substitution in the integral to be $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$; thus the sum is telescoping and equals 1.

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6. An $n \times n$ symmetric matrix $\mathbf{P}$ for which $\mathbf{P}^2 = \mathbf{P}$ is called a *projection matrix*.

(a) Show that if $\mathbf{P}$ is a projection matrix then all its eigenvalues are either 0 or 1.

(b) Let $\mathbf{P}$ be an $n \times n$ projection matrix which has rank r. Show that exactly r of $\mathbf{P}$'s eigenvalues are 1 and exactly $n - r$ are 0.

(c) Let $\vec{u} \in \mathbb{R}^n$ be a unit vector. Define $A = \vec{u}\vec{u}^T$.

i. Determine whether or not A is necessarily a projection matrix. Justify your answer.

ii. Find an eigenvalue of A and a corresponding eigenvector, justifying your claim.

Solution

(a) (3 pts) Let λ be an eigenvalue. Then there is $\vec{v} \neq \vec{0}$ such that $\mathbf{P}\vec{v} = \lambda\vec{v}$. Then $\lambda\vec{v} = \mathbf{P}\vec{v} = \mathbf{P}^2\vec{v} = \lambda^2\vec{v}$ from which $\lambda = 0$ or $\lambda = 1$ since $\vec{v} \neq \vec{0}$.

(b) (3 pts) Since $\mathbf{P}$ is symmetric, it is diagonalizable and so the dimensions of the eigenspaces equal the multiplicity of the corresponding eigenvalues. In particular, the dimension of the null space is $n - r$ by the Rank Theorem, and thus the multiplicity of 0 is $n - r$, leaving multiplicity r for the eigenvalue 1.

(c) (i.) (2 pts) $A^2 = (\vec{u}\vec{u}^T)(\vec{u}\vec{u}^T) = \vec{u}(\vec{u}^T\vec{u})\vec{u}^T = \vec{u}\vec{u}^T = A$ since $\vec{u}^T\vec{u} = \|\vec{u}\| = 1$. Thus A is a projection matrix.

(ii.) (2 pts) Since $A\vec{u} = \vec{u}\vec{u}^T\vec{u} = \vec{u}\|\vec{u}\| = \vec{u}$, we see that $\vec{u}$ is an eigenvector for the eigenvalue $\lambda = 1$. Also, if $\vec{v}$ is any vector orthogonal to $\vec{u}$, then $A\vec{v} = \vec{u}\vec{u}^T\vec{v} = \vec{0}$ since $\vec{u}^T = 0$, so $\vec{v}$ is an eigenvector for the eigenvalue $\lambda = 0$.

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7. For a linear transformation $\mathbf{A}$ on a vector space V, a subspace W of V is called $\mathbf{A}$-invariant if $\mathbf{A}W \subset W$, i.e., for any vector $\vec{w} \in W$, $\mathbf{A}\vec{w} \in W$.

   (a) Consider a linear transformation on $\mathbb{R}^4$ with standard matrix $\mathbf{A}$ under the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & -2 \\ 2 & -1 & 0 & 1 \\ 2 & -1 & -1 & 2 \end{bmatrix}.$$

   Verify that the subpace $W = \operatorname{span}\big\{\vec{v}_1 + 2\vec{v}_2, \vec{v}_2 + \vec{v}_3 + 2\vec{v}_4\big\}$ is $\mathbf{A}$-invariant.

   (b) Now consider a linear transformation $\mathbf{K}$ on $\mathbb{R}^n$ under the basis $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n\}$,

$$\mathbf{K} = \begin{bmatrix} k & 1 & 0 & \cdots & 0 & 0 \\ 0 & k & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k & 1 \\ 0 & 0 & 0 & \cdots & 0 & k \end{bmatrix},$$

   where k is a scalar. Show that:

   i. if some $\mathbf{K}$-invariant subspace W contains $\vec{u}_n$, then $W = \mathbb{R}^n$;

   ii. $\vec{u}_1$ belongs to *any* non-trivial $\mathbf{K}$-invariant subspace of $\mathbb{R}^n$.

Sketch of solution:

(a) (2 pts) Direct calculation: $A(\vec{v}_1 + 2\vec{v}_2) = (\vec{v}_1 + 2\vec{v}_2)$ and $A(\vec{v}_2 + \vec{v}_3 + 2\vec{v}_4) = (\vec{v}_2 + \vec{v}_3 + 2\vec{v}_4)$.

(b) (i.) (4 pts) Calculate $\mathbf{K}\vec{u}_n$ to get $\vec{u}_{n-1} + k\vec{u}_n \in W$, so $\vec{u}_{n-1} \in W$, and so on....

(ii.) (4 pts) For any non-trivial $\mathbf{K}$-invariant subspace W, choose $\vec{u} \in W$ and $\vec{u} \neq \vec{0}$. Let $\vec{u} = \sum_{j=1}^{l} c_j\vec{u}_j$ with $c_l \neq 0$ for some integer $1 \leq l \leq n$. Calculate $\mathbf{K}\vec{u}$ to obtain that $\sum_{j=2}^{l} c_j\vec{u}_{j-1} \in W$, and so on....

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8. Let E be the ellipsoid given by the equation $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 1$. Let B be a rectangular box, centered at the origin with sides parallel to the coordinate axes, of dimensions $l \times w \times h$. Find the dimensions of the box of maximal volume which fits within the ellipsoid E.

Solution

(3 pts) Setting up the correct constrained problem

(4 pts) Deriving the correct equations for Lagrange's theorem

(3 pts) Solving the equations to obtain the correct dimensions.

We must maximize $V = lwh$ subject to the constraint $\frac{l^2}{16} + \frac{w^2}{4} + \frac{h^2}{36} = 1$. Setting up the equations for a Lagrange multiplier we obtain

$$\lambda = \frac{8wh}{l} = \frac{2lh}{w} = \frac{18lw}{h}.$$

Solving for, e.g. l^2, and plugging into the constraint, we obtain $3\frac{l^2}{16} = 1$, so $l = \frac{4}{\sqrt{3}}$. From this, $w = \frac{1}{\sqrt{3}}$ and $h = \frac{9}{\sqrt{3}}$.

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9. Let E be the ellipsoid $x^2 + 16y^2 + z^2 = 16$, and P be the plane $x - 2y - z = 3$, which does not intersect E. Find the point on E which is closest to P, and the (perpendicular) distance from p to P. (Hint: at p the tangent plane to E must be parallel to P.)

Solution

(5 pts) Finding the point(s) where the normal vector to the surface is parallel to the normal vector to the given plane.

(5 pts) Finding the distance from the point on the surface to the plane.

Express E as $g(x, y, z) = 0$ where $g(x, y, z) = x^2 + 16y^2 + z^2 - 16 = 0$. The normal vector to E at a point on E is thus $\nabla g = (2x, 32y, 2z)$. Thus we seek (x, y, z) so that $(2x, 32y, 2z) = \alpha(1, -2, -1)$ for some α and so that $x^2 + 16y^2 + z^2 = 16$. Thus $\frac{\alpha^2}{4} + \frac{\alpha^2}{16} + \frac{\alpha^2}{4} = 16$ which implies $\alpha = \pm\frac{16}{3}$. With $\alpha = \frac{16}{3}$ we obtain the point $p_+ = (\frac{8}{3}, -\frac{1}{3}, -\frac{8}{3})$, and with $\alpha = -\frac{16}{3}$ we obtain the point $p_- = (-\frac{8}{3}, \frac{1}{3}, \frac{8}{3})$. To find the distance from such a point to the plane P we compute $\left|\vec{v} \cdot \frac{\vec{n}}{|\vec{n}|}\right|$ where $\vec{n}$ is a normal vector to P and $\vec{v}$ is a vector from $p_\pm$ to a point on P, say $(3, 0, 0)$. We obtain

$$\left|\left(\frac{8}{3} - 3, -\frac{1}{3}, -\frac{8}{3}\right) \cdot \frac{(1, -2, -1)}{\sqrt{6}}\right| = \frac{3}{\sqrt{6}}, \quad \text{and} \quad \left|\left(-\frac{8}{3} - 3, \frac{1}{3}, \frac{8}{3}\right) \cdot \frac{(1, -2, -1)}{\sqrt{6}}\right| = \frac{9}{\sqrt{6}}.$$

Thus the closest point is $p_+ = (\frac{8}{3}, -\frac{1}{3}, -\frac{8}{3})$ and the distance is $\frac{3}{\sqrt{6}}$.

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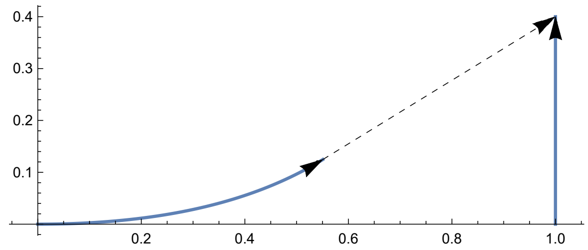
10. Consider a fox chasing a rabbit; the fox's path is $F(t) = (x(t), y(t))$ with $F(0) = (0, 0)$, and the rabbit's path is $(1, t)$. The fox always runs *directly toward* the rabbit at a speed that is twice the distance from the fox to the rabbit; that is, at all time t, the velocity vector of the fox is directly toward the position of the rabbit at that time and its length is twice the distance from the fox to the rabbit. Use this information to find the path of the fox, $F(t)$. Does the fox ever catch the rabbit? If so, when? If not, explain.



Solution

(2 pts) Setting up the pair of ODEs.

(5 pts) Solving the IVPs.

(3 pts) Showing the fox never catches the rabbit.

$(x'(t), y'(t)) = c(1-x(t), t-y(t))$ for some c (potentially depending on t). Using the fact that the speed is always twice the distance, we see that $c = 2$. Thus $x'(t) = 2(1-x(t))$, $x(0) = 0$, and so $x(t) = 1-e^{-2t}$. And $y'(t) = 2(t-y(t))$, $y(0) = 0$, so $y(t) = t - \frac{1}{2} + \frac{1}{2}e^{-2t}$. The distance (squared) between the fox and the rabbit is $d^2(t) = e^{-4t} + \frac{1}{4}(1-e^{-2t})^2$. As $t \to \infty$, $d \to \frac{1}{2}$ and the fox never catches the rabbit – the closer the fox gets to the rabbit, the slower the fox runs.