Calculators are allowed but you must give exact values, not approximate answers.
You must clearly justify all answers.
Problems are of equal weight.

Number: ______________________________

1. Find the value of a for which the integral

$$\int_1^\infty \frac{a}{x(2x+a)}dx$$

converges to the value of 1.

Solution:

$$\int_1^\infty \frac{a}{x\,(2x+a)}\,dx = \int_1^\infty \left(\frac{1}{x} - \frac{2}{2x+a}\right)dx = \ln\left(\frac{x}{2x+a}\right)\Bigg|_1^\infty$$

$$= \ln\left(\frac{2+a}{2}\right) \overset{set}{=} 1$$

Thus, $a = 2(e-1)$.

2. a. Show, by an example, that linear dependence of the columns of a matrix does not imply the linear dependence of the rows. (Note: you need to *briefly* indicate why the columns of your matrix are linearly dependent and the rows are not. A proof is not needed.)

   b. State the rank-nullity theorem, a.k.a. the dimension theorem, for matrices.

   c. Show that, if an $n \times n$ matrix A is such that $A^2 = A$, then $\text{rank}(A) + \text{rank}(I - A) = n$.

Solution: (a) An example is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

It must be a non-square matrix.

(b) The Dimension Theorem: For any $(m \times n)$ matrix A,

$$rank(A) + nullity(A) = n$$

where $nullity(A)$ is the dimension of the solution space of $Ax = 0$.

(c) Let $K = \{x : Ax = 0\}$, the kernel of A, and $S = \{(I - A)x : x \in \mathcal{R}^n\}$, the column space of $I - A$. If we can show $K = S$, then $dim(K) = dim(S)$, i.e., $nullity(A) = rank(I - A)$, and so the result follows from the dimension theorem. Let $x \in K$. Note that $x \in \mathcal{R}^n$, and also $x = x - 0 = x - Ax = (I - A)x \in S$. Thus, $K \subset S$. Let $y \in S$. Then $y = (I - A)x$ for some $x \in \mathcal{R}^n$. This implies that $Ay = A(I - A)x = (A - A^2)x = 0$ by hypothesis, i.e., $y \in K$. Thus, $S \subset K$, and $S = K$.

3. Find the real-valued function function $y : \mathbb{R} \to \mathbb{R}$ which satisfies

$$\begin{aligned} 2y'' - y' - 6y &= 0, \\ y(0) &= 1, \\ y'(0) &= 0. \end{aligned}$$

Solution: The characteristic equation for this ODE is $2r^2 - r - 6 = 0$ which has roots at -1.5 and 2. So solutions to the ODE are of the form $y(t) = c_1 e^{-3t/2} + c_2 e^{2t}$. The initial values say:

$$\begin{aligned} c_1 + c_2 &= 1 \\ -1.5c_1 + 2c_2 &= 0. \end{aligned}$$

So $c_1 = 4/7$ and $c_2 = 3/7$.

4. Let M denote the $n \times n$ matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Note that $M^2 = nM$.

a. What are the possible eigenvalues of $aI_n + bM$? Here a and b are scalars and I_n is the identity $n \times n$ matrix.

b. Let A be the 4×4 matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Let k be an integer greater than 1. Find A^k. Note: You may express your answer in the form of the product $B^x H C^y$, where B, H, C are nontrivial matrices whose elements must be specified.

Solution: (a) Let (x, λ) be an eigenvector-eigenvalue pair of $aI_n + bM$. Then (i) if $\sum_{i=1}^n x_i = 0$, i.e., $Mx = 0$, then $(aI_n + bM)x = \lambda x$ implies that $ax = \lambda x$ and hence $\lambda = a$; (ii) if $\sum_{i=1}^n x_i \neq 0$, i.e., $Mx \neq 0$, then $(aI_n + bM)x = \lambda x \quad \Rightarrow (aM + bM^2)x = \lambda Mx \quad \Rightarrow (aM + nbM)x = \lambda Mx \quad \Rightarrow \quad (a + nb - \lambda)Mx = 0$. Since $Mx \neq 0$, $\lambda = a + nb$.

(b) Note that $A = aI_4 + bM$, where $a = -1$ and $b = 1$. So the eigenvalues of A are $\lambda = a = -1$ and $\lambda = a + 4b = 3$ by part (a).

Case 1: $\lambda = -1$. Thus, $(A + I)x = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} x = 0,$

implying that $\sum_{i=1}^n x_i = 0$, consistent with the condition for $\lambda = a$. Thus a basis for x is $\left\{[1, 0, 0, -1]', [0, 1, 0, -1]', [0, 0, 1, -1]'\right\}$.

Case 2: $\lambda = 3$. Thus, $(A - 3I)x = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix} x = 0.$

By inspection, or some other method, $x = [1, 1, 1, 1]'$.

Putting these together, we can write $AP = PD$, where

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Hence, $A^k = PD^kP^{-1}$

5. Let $f_n(x)$ be defined recursively as $f_n(x) = 1 - \int_0^x s[f_{n-1}(s)]ds$ with $f_0(x) = 1$.

   a. Determine $f_1(x), f_2(x), f_3(x), f_4(x)$.

   b. What is the coefficient of the highest order term in f_n?

   c. Show that for each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x)$ exists.

Solution: $f_1(x) = 1 - \frac{x^2}{2}$, $f_2(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8}$, $f_3(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$, $f_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384}$. We can see from this that the lower order terms never change and that f_n is $f_{n-1} + (-1)^n a_n x^{2n}$ where a_n is gotten by noting that that $a_1 = 1/2$, $a_2 = 1/2 * 1/4$, $a_3 = 1/2 * 1/4 * 1/6$,$a_4 = 1/2 * 1/4 * 1/6 * 1/8$. So $a_n = [2^n * (n!)]^{-1}$.

To show the limit exists, we can write $\lim_{n\to\infty} f_n(x)$ as the sum $\sum_{n=0}^{\infty}(-1)^n a_n (x^2)^n$ which converges since, by the Ratio Test (for instance):

$$|\frac{a_{n+1}(x^2)^{n+1}}{a_n(x^2)^n}| = x^2(2^{-1})(n+1)^{-1}$$

which converges to 0 for each x.

6. A curve is given by the parametric equations

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 < t < \frac{\pi}{2}.$$

   a. Show that, near $t = 0$, the equations can be approximated by

$$a - x \approx 3at^2/2, \quad y \approx at^3.$$

   b. Find the length of the curve. *Note:* A useful identity is $\sin 2\theta = 2\sin\theta\cos\theta$.

Solution: (a) By Maclaurin series, $f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f"(0)}{1!}x^2$ if x is close to 0. Thus, $\sin t \approx t$ and $\cos t \approx 1 - \frac{1}{2}t^2$. It follows then

$$\begin{aligned} x &= a\cos^3 t \approx a\left(1 - \frac{1}{2}t^2\right)^3 = a\left(1 - \frac{3}{2}t^2 + \cdots\right) \\ &\approx a - \frac{3}{2}at^2 \end{aligned}$$

Thus, $a - x \approx \frac{3}{2}at^2$ and $y = a\sin^3 t \approx at^3$.

(b) Since $(\frac{dx}{dt})^2 = 9a^2\cos^4 t\sin^2 t$ and $(\frac{dy}{dt})^2 = 9a^2\sin^4 t\cos^2 t$, the expression for the length s of a section of curve in parametric form is

$$\begin{aligned} \left(\frac{ds}{dt}\right)^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\ &= 9a^2\sin^2 t\cos^2 t\left(\cos^2 t + \sin^2 t\right) \\ &= 9a^2\sin^2 t\cos^2 t. \end{aligned}$$

Thus, $\frac{ds}{dt} = 3a\sin t\cos t = \frac{3}{2}a\sin(2t)$, and $s = \int_0^{\pi/2}\frac{3}{2}a\sin(2t) = \frac{3}{2}a$.

7. A company is looking to invest in two assets. If w_1 and w_2 are the amounts of the investment budget invested in each asset, then the variance of the total investments is given by

$$\frac{1}{4}w_1^2 + \frac{1}{9}w_2^2 + \frac{1}{3}\rho w_1 w_2$$

where $\rho \in [-1, 1]$ is a known fixed number. The expected return of this investment is given by the expression

$$8w_1 + 4w_2$$

What is the best choice of w_1 and w_2—in terms of ρ—to minimize the variance while ensuring an expected return of 24?

Solution: The Lagrange multiplier rule says we need to solve

$$\begin{aligned} 2w_1 + w_2 &= 6 \\ 8\lambda + \frac{1}{2}w_1 + \frac{\rho}{3}w_2 &= 0 \\ 4\lambda + \frac{2}{9}w_2 + \frac{\rho}{3}w_1 &= 0. \end{aligned}$$

This has the solution $(w_1, w_2, \lambda) = \frac{1}{24\rho-25}(12(3\rho-4), 18(4\rho-3), 3-3\rho^2)$.

8. a. Prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\lim_{n\to\infty} n^\alpha a_n = A$ for some $\alpha > 1$, where A is some real number.
   b. For each of the following, determine whether the series converges or diverges and provide the justification of your answer.
      (ii) $\sum_{n=1}^{\infty} \sin(\frac{\pi}{n^2})$
      (ii) $\sum_{n=1}^{\infty} \frac{(n+1)!}{n^n}$.

Solution: (a) Let $\epsilon > 0$. By hypothesis $\exists N$ such that $\forall n \geq N$, $|n^\alpha a_n - A| \leq \epsilon$. This implies $-\epsilon \leq n^\alpha a_n - A \leq \epsilon$ and so $\frac{A-\epsilon}{n^\alpha} \leq a_n \leq \frac{A+\epsilon}{n^\alpha}$. It follows then $|a_n| \leq \frac{r}{n^\alpha}$, where $r = \max\{|A-\epsilon|, |A+\epsilon|\}$. Thus, $\sum_{n=N}^{\infty} |a_n| \leq r\sum_{n=N}^{\infty} \frac{1}{n^\alpha} < \infty$ when $\alpha > 1$. This shows that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) (i) Note that

$$\lim_{n\to\infty} n^2 \sin\left(\frac{\pi}{n^2}\right) = \lim_{n\to\infty} \frac{\sin(\pi/n^2)}{1/n^2} = \pi,$$

by L'hopital's rule. By part (a), $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n^2}\right)$ converges absolutely and hence converges.

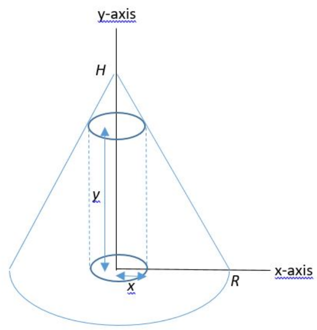
(ii) Note that

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+2)!/(n+1)^{n+1}}{(n+1)!/n^n} = \lim_{n\to\infty}\left[\frac{n+2}{n+1}\frac{n^n}{(n+1)^n}\right] \\ &= \lim_{n\to\infty}\left[\frac{n+2}{n+1}\left(1+\frac{1}{n}\right)^{-n}\right] = e^{-1} < 1 \end{aligned}$$

Thus, by the ratio test, $\sum_{n=1}^{\infty} a_n$ converges.

9. A solid cone of uniform density ρ, height H and base radius R rotates at constant angular velocity ω about its axis. Find the kinetic energy of the cone. Note: The kinetic energy of a particle of mass m moving at speed v is given by $E = \frac{1}{2}mv^2$.

Solution: Based on the diagram below,



For a cylindrical shell of thickness dx, the amount of kinetic energy is $dE = \frac{1}{2}(\omega x)^2 dm = \frac{1}{2}\omega^2 \rho x^2 \cdot 2\pi xy dx$. Using the fact that $\frac{y}{R-x} = \frac{H}{R}$,

$$\begin{aligned} E &= \pi\omega^2\rho \int_0^R x^3 y \, dx = \pi\omega^2\rho \frac{H}{R} \int_0^R x^3(R-x) \, dx \\ &= \pi\omega^2\rho \frac{HR^4}{20}. \end{aligned}$$

10. Given a position x, y, the elevation in a portion of a canyon floor is given by

$$f(x,y) = (1-x)^2 + 2(y-x^2)^2$$

We drive a vertical post into the ground at $P_1 = (0,0)$, $P_2 = (1,0)$ and at $P_3 = (1,1)$. The portion of each post not buried in the ground is length 1. A (not necessarily level) platform is attached to the top of these three posts. If we put a fourth post at P_2 which is normal to the canyon floor with length 1 sticking out of the ground, what is the exact angle between this fourth post and the upward normal of the platform?

Solution: The platform is given by the three points $(0,0,2), (1,0,3), (1,1,1)$, so the platform's normal given by $[(0,0,2) - (1,0,3)] \times [(1,1,1) - (1,0,3)] = (-1,0,-1) \times (0,1,-2) = \begin{vmatrix} i & j & k \\ -1 & 0 & -1 \\ 0 & 1 & -2 \end{vmatrix} = 1i - 2j - k$. So the upward normal of the platform is $(-1,2,1)$. $\nabla f(x,y) = [-2(1-x) - 8x(y-x^2), 4(y-x^2)]^T$. So setting both of those values to zero, leads us to conclude $y = x^2$ (from the second) and $x = 1$ from the first. The normal at $(1, 0))$ is of course $(8, -4, 1)$. Thus the dot product of the two normal vectors is -15. Thus the angle is $\cos^{-1}(\frac{-15}{\sqrt{81}\sqrt{6}}) = \cos^{-1}(\frac{-5}{3\sqrt{6}})$.