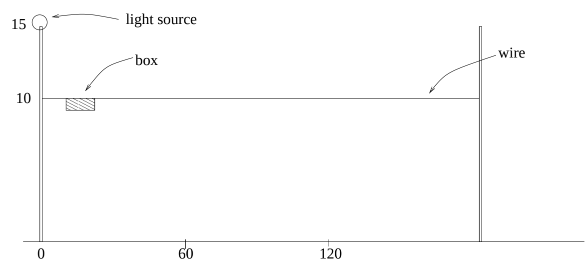
Graduate Qualifying Exam Solutions– Fall 2019

Problem 1. A wire is stretched, at a height of 10 feet, between two poles that are more than 120 feet apart. A box will be pulled along the wire from the start-pole to the finish-pole. A light source is placed on the start-pole at a height of 15 feet. A photosensitive plate sits on the ground, beneath the wire, at a distance of 60 feet from the start-pole. See the figure.



The box starts with its leading edge directly below the light source. At time $t = 0$, the box starts moving, and will accelerate at a constant rate of 10 feet per second per second until the shadow of the box hits the photosensitive plate. At the moment the shadow hits the plate, the acceleration stops and the box will maintain a constant velocity until it crashes into the finish-pole.

At what time will the leading edge of the box reach a point 120 feet from its starting position? What will be the velocity of the box at that point?

Solution: Let x_b be the distance of the leading edge of the box from its start position and let x_s be the position of the leading edge of the box's shadow along the ground. By similar triangles, $x_s = 3x_b$. Hence, the shadow hits the plate when $x_b = 20$.

Timing of shadow hitting plate: $\frac{d^2x_b}{dt^2} = a = 10$, so $\frac{dx_b}{dt} = v_0 + at = 0 + 10t = 10t$, and, finally, $x_b = 5t^2$. Then, since the shadow hits the plate when $x_b = 20$, we have $20 = 5t^2$, so $t = 2$. The shadow hits the plate at a time 2 seconds after the box starts moving. At that moment, its velocity is $\frac{dx_b}{dt} = 10t = 10(2) = 20$feet per second.

The box then goes the remaining 120 - 20 = 100 feet at a rate of 20 feet per second. This takes 5 seconds. The total time is then 2+5 = 7 seconds. Since there has been no acceleration since the shadow hit the photosensitive plate, the velocity is still 20 feet per second.

Problem 2. A 1-unit-long rod lies on the x-axis with its left end at the origin. Its density is given by

$$f(x) = \begin{cases} \sin(\pi x), & 0 \le x < \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \le x < 1. \end{cases}$$

(a) Find the total mass of the rod.

Solution: Let M be the total mass of the rod. Then

$$\begin{aligned} M &= \int_0^1 f(x)dx = \int_0^{1/2} \sin(\pi x)dx + \int_{1/2}^1 (2-2x)dx = \frac{-\cos(\pi x)}{\pi}\Big|_0^{1/2} + \left(2x - x^2\right)\Big|_{1/2}^1 \\ &= \frac{1}{\pi} + \left(1 - \left(1 - \frac{1}{4}\right)\right) = \frac{1}{\pi} + \frac{1}{4} = \frac{4+\pi}{4\pi}. \end{aligned}$$

(b) Suppose the rod is cut so that only the portion between $x = 0$ and $x = \frac{1}{2}$ remains. We want to place a fulcrum under the rod so that the rod perfectly balances on the fulcrum. At what position (x) should we place the fulcrum?

Solution: This is just asking for the center of mass of the rod. Let $\bar{x}$ be the center of mass. From the above calculation, I see that the mass of this part of the rod is just $\frac{1}{\pi}$. Then

$$\begin{aligned}\bar{x} &= \frac{\int_0^{1/2} xf(x)dx}{\int_0^{1/2} f(x)dx} = \pi \int_0^{1/1} xf(x)dx = \int_0^{1/2} x\sin(\pi x)dx \\ &\overset{a}{=} \pi \left[\frac{-x\cos(\pi x)}{\pi}\Big|_0^{1/2} - \int_0^{1/2} \frac{-\cos(\pi x)}{\pi} dx \right] = \pi \left[0 + \frac{\sin(\pi x)}{\pi^2}\Big|_0^{1/2} \right] = \pi \cdot \frac{1}{\pi^2} = \frac{1}{\pi}.\end{aligned}$$

[a]Integration by parts using $u = x$ and $dv = \sin(\pi x)$.

Problem 3. Let M be the vector space of 2×2 matrices with real entries and let $\vec{v_0} \in \mathbb{R}^2$ be a nonzero vector. Define the set S by

$$S = \{A \in M : \; A\vec{v_0} = \vec{0}\}$$

a) Prove that S is a subspace of M.

Solution: The 2×2 zero matrix belongs to S, so S is nonempty. Suppose A and B belong to S. Then $(A+B)\vec{v_0} = A\vec{v_0} + B\vec{v_0} = \vec{0}$ and for any $k \in \mathbb{R}$, $kA\vec{v_0} = k\vec{0} = \vec{0}$. Therefore S is closed under addition and scalar multiplication so S is a subspace of M.

b) Find a basis for S.

Solution: Without loss of generality, suppose that $\vec{v_0} = (x_0, y_0)$ where $y_0 \neq 0$. Let $A \in S$ be given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

By writing the system of equations determined by $A\vec{v_0} = \vec{0}$ and simplifying we get $b = -ax_0/y_0$ and $d = -cx_0/y_0$. Therefore, b is determined by a and d is determined by c and S has dimension 2. Any two linearly independent matrices in S will then be a basis. One possible basis is determined by taking $a = 1$ and $c = 0$ and then $a = 0$ and $c = 1$ to get

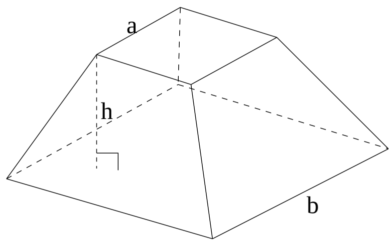
$$\begin{bmatrix} 1 & -x_0/y_0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & -x_0/y_0 \end{bmatrix}$$

c) Prove or disprove: For all $\lambda \in \mathbb{R}$, there exists $A \in S$ such that λ is an eigenvalue of A.

Solution: The claim is true. Consider the first element of the basis in the last part. This matrix is upper triangular and its eigenvalues are 1 and 0. By scaling this matrix by a factor of λ, we obtain another upper triangular element of S, now with eigenvalues λ and 0.

Problem 4. The ancient Egyptians would model a pile of grain as having the shape of a truncated 4-sided pyramid, which we will call a frustum. They knew that such a frustum, whose base was a square of side-length b, whose height was h, and whose top was a square of side-length a, would have a volume of $V = \frac{h}{3}\left(a^2 + ab + b^2\right)$.

A quick sketch of such a frustum is in the figure below. The frustum in the figure is not to scale for this problem, but is included just to give you the idea that this type of frustum is just a pyramid with the top chopped off.



Frustum of a Pyramid

For this particular problem, consider a frustum to which grain is being added at a rate of 1 cubic cubit per minute. As the volume of the grain pile increases, imagine that it retains the shape of a frustum for which a, b, and h are increasing while they maintain a constant ratio of $a : b : h = 1 : 2 : 1$. How fast is the height of the grain pile changing when the volume of the grain pile is 504 cubic cubits?

Solution: For this frustum, $V = \frac{h}{3}(a^2 + ab + b^2) = \frac{h}{3}\left(h^2 + 2h^2 + 4h^2\right) = \frac{7}{3}h^3$, so $\frac{dV}{dh} = 7h^2$.

When $V = 504$, we have $504 = \frac{7}{3}h^3$, so $h = \sqrt[3]{216} = 6$. At this moment, we have $\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{dh}{dV} = 2 \cdot \frac{1}{7 \cdot 6^2} = \frac{1}{126}$. The height of the grain pile is changing at a rate of $1/126$ cubits per minute.

Problem 5. Suppose A is a 3×4 matrix with real entries and $\vec{b}$ is a vector such that the augmented matrix $[A \,|\, \vec{b}]$ has rank 3.

a) Is it possible for the system of equations $A\vec{x} = \vec{b}$ to be inconsistent? Justify your answer.

Solution: Yes. This will happen if the rank of A is 2 and $\vec{b}$ is not in Col A. The row reduced echelon form of $[A \,|\, \vec{b}]$ would have a pivot in the last column.

b) Is it possible for the system of equations $A\vec{x} = \vec{b}$ to have a unique solution? Justify your answer.

Solution: No. If the system is consistent, the last column of the augmented matrix is not a pivot column. Since the rank of the augmented matrix is 3, every row of A must have a pivot so that there is a column of A that is not a pivot column so the system has a free variable, and hence infinitely many solutions.

Parts c) and d): Repeat parts a) and b) where now A is a 4×3 real matrix, and A itself (not $[A \,|\, \vec{b}]$) has rank 3.

Solutions: Yes, the system can be inconsistent. The rank of A is the dimension of the range of the transformation $\vec{x} \to A\vec{x}$. This transformation maps $\mathbb{R}^3$ into $\mathbb{R}^4$. Since the codomain has dimension 4 and the range dimension 3, this transformation is not onto. Choosing the target vector $\vec{b}$ outside Col A will result in an inconsistent system. When $A\vec{x} = \vec{b}$ is consistent, it will have a unique solution. Each column of A is a pivot column, so the system cannot have a free variable, so whenever solutions exist, they are unique.

Problem 6. Give an example of a function $f : \mathbb{R}^3 \to \mathbb{R}$ with all of the following properties (along with an explanation why your function satisfies the required properties), or explain why there is no such function:

- f is defined and differentiable at all $(x, y, z) \in \mathbb{R}^3$
- $f(x, y, z) \geq 0$ for all $(x, y, z) \in \mathbb{R}^3$
- f has a local maximum at the origin and $f(0, 0, 0) = 1$
- f is *radial* meaning that f is constant on each sphere centered at the origin, ie for any $r > 0$, there exists a real constant c_r such that $f(x, y, z) = c_r$ for all (x, y, z) on the sphere centered at the origin with radius r.
- $f(x, y, z) = 0$ if and only if (x, y, z) is on the sphere of radius 2 centered at the origin.

Solution: This is possible. A radial function can be written in the form $f(x, y, z) = F(R)$ where $R = x^2 + y^2 + z^2$. If F is differentiable, say a polynomial, and $F'(0) = 0$, f will be differentiable ($F'(0) = 0$ is necessary for differentiability at the origin). To have a minimum on the sphere of radius 2, we need $F'(4) = 0$. If $F' = AR(R - 4)$ for $A > 0$, $f(x, y, z) = F(R)$ will be differentiable, have a local maximum at $R = 0$ and a global minimum when $R = 4$. Integrating to recover F, we get $F(R) = (1/3)AR^3 - 2AR^2 + C$. We now choose A and C so that $F(0) = 1$ and $F(4) = 0$. We get $A = 3/32$ and $C = 1$. Finally, we obtain:

$$f(x, y, z) = (1/32)(x^2 + y^2 + z^2)^3 - (3/16)(x^2 + y^2 + z^2)^2 + 1$$

Many other answers are possible.

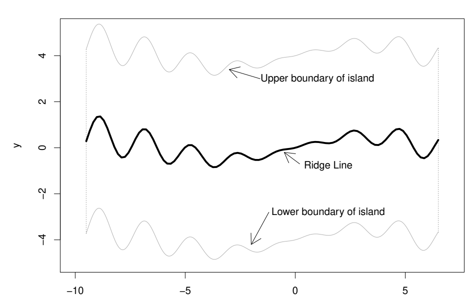
Problem 7. An island has the form of a hill that rises out of the sea. The island has a ridge-line given by $f(x)$. As one walks along the ridge-line, the island slopes downward on either side. Phil has purchased the slice of the island that lies above sea level between $x = -9.5$ and $x = 6.5$.

The height above sea level, H, of Phil's piece of this island, at any point (x, y), is given by the function

$$H(x, y) = 4\left(1 - \frac{(2x + 3)^4}{16^4}\right)\left(1 - \frac{(y - f(x))^2}{16}\right),$$

for all points (x, y) for which $H(x, y) > 0$ and $-9.5 < x < 6.5$.

The figure below illustrates the shape of Phil's piece of the island, as seen from above, including the ridgeline ($f(x)$, dark curve), and the upper and lower boundaries of the island (light curves). Above the "Upper Boundary" curve, and below the "Lower Boundary" curve in the figure is the ocean. To the left and right of the vertical dotted lines are land that Phil doesn't own.



Use a change of variables with $u = 2x + 3$ and $v = y - f(x)$ to find the volume of the piece of the island that Phil owns and which is above sea level.

Solution: Inverse transform: $x = \frac{u-3}{2}$ and $y = v + f(x) = v + f\left(\frac{u-3}{2}\right)$. Then $|J| = \left| det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right| = \left| det \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} f'\left(\frac{u-3}{2}\right) & 1 \end{bmatrix} \right| = \frac{1}{2}$.

Boundaries: $-9.5 < x < 6.5$ yields $-16 < u < 16$. For these values of u, we have $2\left(1 - \frac{u^4}{16^4}\right) > 0$, so $\tilde{H}$ will be positive provided $\left(1 - \frac{v^2}{16}\right) > 0$. Our bounds are then the rectangle $-16 < u < 16$, $-4 < v < 4$.

Finding the volume: $V = \int_{-16}^{16} \int_{-4}^{4} 2\left(1 - \frac{u^4}{16^4}\right)\left(1 - \frac{v^2}{16}\right) dv du = \int_{-16}^{16} 2\left(1 - \frac{u^4}{16^4}\right)\left(v - \frac{v^3}{3 \cdot 16}\right)\Big|_{-4}^{4} du$
$= \int_{-16}^{16} 2\left(1 - \frac{u^4}{16^4}\right)\left(4 - \frac{4^3}{3 \cdot 16} - \left(-4 - \frac{-4^3}{3 \cdot 16}\right)\right) du = \int_{-16}^{16} \frac{32}{3}\left(1 - \frac{u^4}{16^4}\right) du = \frac{32}{3}\left(u - \frac{u^5}{5 \cdot 16^4}\right)\Big|_{-16}^{16}$
$= \frac{32}{3}\left(16 - \frac{16^5}{5 \cdot 16^4} - \left(-16 - \frac{-16^5}{5 \cdot 16^4}\right)\right) = \frac{4096}{15}$.

Problem 8. Consider the differential equation $y' = y^\alpha$ where α is a positive parameter, and assume that $y \geq 0$ to avoid complications that arise for some α when $y < 0$.

a) Find the general solution.

Solution: This equation is separable. Separating, integrating and simplifying, we get $y(t) = [(1-\alpha)(t + C)]^{1/(1-\alpha)}$, unless $\alpha = 1$, in which case we get $y(t) = Ce^t$. Note that $y(t) = 0$ is also a solution for any α.
For the remaining parts of the question, let $y_\alpha(t)$ be the solution to the initial value problem for the given equation with initial condition $y(0) = 1$.

b) Find all values of α for which y_α is defined for all $t > 0$ or explain why no such α exist.

Solution: If $(1-\alpha) > 0$, ie $\alpha < 1$, y_α is defined for all $t > 0$. The same is true when $\alpha = 1$. This can be seen from the formulas for y_α obtained from part a): $y_\alpha(t) = [(1-\alpha)t + 1]^{1/(1-\alpha)}$, unless $\alpha = 1$, in which case $y_\alpha(t) = e^t$. If $\alpha > 1$, $(1-\alpha)t + 1$ will eventually become zero, at which point the solution becomes undefined (note that the exponent is negative). The graph will have a vertical asymptote at this particular time, approaching infinity from the left. So the answer is $\alpha \in (0, 1]$.

c) Find all values of α for which y_α is bounded on $[0, \infty)$ or explain why no such α exist.

Solution: There are no such α. The only possible choices based on part c) are $\alpha \in (0, 1]$. The formula in part b) shows that for any α in this range the solution will increase without bound, as it is either exponential, or the composition of an increasing linear function with a (positive) power function.

Problem 9. a) Recall that the series $\sum_{j=1}^{\infty} 1/j$ diverges. Prove that for all $M \in \mathbb{R}$ and for all $t > 0$, there exists s such that $\sum_{j=t}^{s}(1/j) > M$.

Solution: The harmonic series is divergent and has terms which are all positive. Thus, the sequence of partial sums (from any starting point t) is increasing and not bounded above. Thus given any M and t, the claim holds.

b) Use part a) (or find some other means) to provide an example of a divergent series $\sum_{n=1}^{\infty} a_n$ such that $a_n \to 0$ as $n \to \infty$ and such that the series has infinitely many positive and infinitely many negative terms. **Note:** An explicit formula for a_n is **not** required; it is sufficient to give a detailed description of how to construct the terms.

Solution: We will construct such a series by choosing a_n to be $\pm 1/n$ for each n. Thus $a_n \to 0$ as $n \to \infty$. Let S_n denote the sequence of partial sums for our series. Infinitely many of the S_n will be greater than or equal to 1 and infinitely many will be less than or equal to zero. This means that S_n cannot converge so the series will diverge. We start with $a_1 = 1$, so that $S_1 = 1$. We now choose $a_2 = -1/2$, $a_3 = -1/3$, $a_4 = -1/4$. Then $S_4 < 0$. Take $a_5 = 1/5$ and continue choosing $a_n = 1/n$ until $S_n \geq 1$. This is possible by part a). Then let $a_{n+1} = -1/(n+1)$ and continue choosing negative terms until the partial sum becomes nonpositive, again possible by part a). Repeat this process, choosing positive terms from the harmonic series until the partial sum is at least 1 and then choosing terms from the negative harmonic series (in order) until the partial sums become negative.

Problem 10. For each of the series below, find all values of p for which the series converges and all values of p for which it diverges. Justify your claims, including a justification for the use of any convergence tests.

$$a) \sum_{n=2}^{\infty} \frac{(\log n)^p}{n}$$

Solution: We use the integral test (this requires justification: $f(x) = (\log x)^p/x$ is decreasing when $\log x > p$). Using the substitution $u = \log x$, the integral $\int_2^\infty (\log x)^p/x \, dx = \int_{\log 2}^\infty u^p \, du$. This integral converges when $p < -1$ and diverges when $p \geq -1$. Comparison test can also be used to show divergence for $p \geq 0$.

$$b) \sum_{n=1}^{\infty} \frac{\log n}{n^p}$$

Solution: If $p \leq 1$, the series diverges either because the terms do not approach zero (when $p \leq 0$) or by the comparison test, using the series $\sum_{n=1}^\infty \frac{1}{n^p}$. Now suppose $p > 1$. Choose $0 < r < p - 1$. Then for all n sufficiently large $\log n < n^r$ (which can be seen by l'Hopital), so that, for such n, $(\log n)/n^p < n^{r-p}$. We know that $r - p < -1$, so that the given series converges by the comparison test.