Graduate Qualifying Exam – Spring 2018

Note: You have *three hours* to complete this exam. Calculators and other electronic devices are *not allowed*. Show all work to receive full credit.

Problem 1. A girl is standing still, watching a bird that is flying horizontally in a straight path at a rate of 30 feet per second, at a height of 300 feet above the girl. The bird passes over the girl and continues along its path. How fast is the distance between the bird and the girl changing when the bird is 500 feet away from the girl?

Problem 2. Consider the region on the x-y plane that is bounded by $y = (x-1)^2$, the x-axis, and the y-axis. Find the volume generated when this region is rotated about the line $x = -1$.

Problem 3. A manufacturer's production is modeled by the Cobb-Douglas function

$$f(x, y) = Kx^{\alpha}y^{1-\alpha},$$

where K and α are constants with $K > 0$ and $0 < \alpha < 1$. In the Cobb-Douglas function, x represents the amount of labor used, and y represents the amount of capital. Each labor unit costs m dollars and each capital unit costs n dollars. Suppose the company can spend only p dollars on the production.

(a) Find the relationship satisfied by m, n, x, y, and p that expresses the idea that the company will spend p dollars on the production.
(b) Find the values of x and y that maximize the production function for this company, in terms of α, m, n, and p.

Problem 4. Consider the following double integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-(y-x)^2-y^2}\,dxdy.$$

(a) Express I as an integral in the new variables $u = x + y$ and $v = -x + y$.

(b) Compute the exact value of I assuming as known that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$.

Problem 5. (a) Find the interval of convergence of the power series $\displaystyle\sum_{n\geq 1} \frac{x^n}{n}$.

(b) Find a closed form representation for $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

(c) Find the exact value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(2018)^n}$.

Problem 6. The sequence $(a_n)_{n\geq 1}$ is defined recursively by $a_1 > 0$ and

$$a_{n+1} = \frac{na_n}{n + a_n^2}, n \geq 1.$$

(a) Show that $(a_n)_{n\geq 1}$ is a bounded and monotonic sequence.
(b) Let $k \geq 1$. Express $\frac{1}{a_{k+1}} - \frac{1}{a_k}$ as a function of the ratio $\frac{a_k}{k}$.
(c) Use parts (a), (b) to show that, if $l = \lim_{n\to\infty} a_n$, then for all $n \geq 1$ we have

$$\frac{1}{a_{n+1}} \geq \frac{1}{a_1} + l \sum_{k=1}^{n} \frac{1}{k}.$$

(d) Compute the exact value of l.

Problem 7. Assume that A, B are (real) matrices of dimensions $m \times n$ and $n \times p$, respectively.
(a) Show that the null space of B is a subset of the null space of AB.
(b) Prove that $\text{rank}(AB) \leq \text{rank}(B)$.
(c) State two square matrices A, B for which the inequality in (b) is strict.

Problem 8. Let A be a 2×2 matrix that satisfies $A^2 + 3A - 2I_2 = O_2$; I_2 and O_2 denote the 2×2 identity, respectively zero, matrices.
(a) Find the possible values of an eigenvalue of A.
(b) Show that A is invertible.
(c) Simplify the expression $B = \left(\frac{1}{2}A + A^{-1}\right)^2 - \frac{13}{4}I_2$. Then, compute the determinant of B.

Problem 9. A large company has total assets of \$3 trillion invested in the U.S., Asia, and Europe. At the start, none of the money is in the U.S., \$2 trillion is in Asia, and \$1 trillion is in Europe. Each year, half of the money in the U.S. stays in the U.S., $\frac{1}{4}$ goes to Asia, and $\frac{1}{4}$ goes to Europe. For each of Asia and Europe, half of the money stays home and half is sent to the U.S.
(a) Find the matrix, A, that gives

$$\begin{bmatrix} \text{U.S.} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{[\text{year } k+1]} = A \begin{bmatrix} \text{U.S.} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{[\text{year } k]}$$

(b) Find the eigenvalues and eigenvectors of A.
(c) Write the vector $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ as a linear combination of the eigenvectors.
(d) If year 0 is the initial year, with no money in the U.S., \$2 trillion in Asia, and \$1 trillion in Europe, find the distribution of the money in year k.

Problem 10. (a) Find the general solution $y(t)$ of the differential equation $5y'' + 10y' + 6y = 0$.
(b) Compute $\lim_{t\to\infty} y(t)$.
(c) Assuming that $y(0) \neq 0$, prove that there is a value $t_0 \in \mathbb{R}$ for which $|y(t_0)| > 1$.
(d) Use parts (b) and (c) to show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(t) = \frac{y^2(t)}{1 + y^4(t)}$ attains a maximum value of 0.5.