Graduate Qualifying Examination Syllabus

Note. Questions frequently involve modeling and applications based on the material below. Candidates should be familiar with the use of mathematical induction.

Single Variable Calculus

Definition and graphical interpretation of derivatives. Relationship between differentiability and continuity. The product, quotient and chain rules. Implicit differentiation. Linear approximation. Optimization using simple mathematical modeling. Related rates. L'Hôpital's rule. Parametric equations.

The definite integral: definition (Riemann sums) and interpretation in terms of area. Antiderivatives. The fundamental theorem of calculus (both parts). Techniques of integration, including integration by substitution and integration by parts. Polynomial long division and partial fractions. Improper integrals. Applications of integration, such as computing area, volume, arc length, total mass, center of mass and total work. Area and arc length in polar coordinates. Simple applications to probability.

Multivariable Calculus

Functions of two and three variables. Graphs of functions of two variables. Contour diagrams. Linear functions.

Vectors in three dimensions: components, magnitude, addition, scalar multiplication.

Partial derivatives: definition, interpretation, visualization and computation. Tangent plane approximation. Differentials. Directional derivatives and the gradient vector, in two and three dimensions. The chain rule. Second order partial derivatives and their use. Differentiability of functions of two variables: relation to partial derivatives.

Local extrema and saddle points. Unconstrained optimization. Method of least squares. Constrained optimization and Lagrange multipliers. Interpretation of Lagrange multipliers.

Double and triple integrals. Area and volume elements in cartesian, polar, cylindrical and spherical coordinates. Change of variables: change of domain and Jacobian.

Parameterized curves: position, velocity and acceleration. Vector fields.

Limits and Infinite Series

Limit of a function at infinity: rigorous definitions. Limit of a function at a point: rigorous definitions. One sided limits. Limit theorems for sums, products, quotients and compositions. Continuity at a point and on an interval.

Infinite sequences: convergence and divergence. Limit theorems for sums, products and quotients. Computation of limits. Convergence of bounded monotonic sequences.

Infinite series: convergence and divergence. Geometric and harmonic series; p-series. The comparison, limit comparison, ratio, nth root and integral tests. Absolute and conditional convergence. The alternating series test.

Power series: radius and interval of convergence. Absolute convergence within interval of convergence. Differentiation and integration of power series. Taylor's theorem, with Cauchy's remainder estimate.

Linear Algebra

Systems of linear equations: augmented matrix representation. Row reduction. Echelon and reduced echelon forms. Existence and uniqueness of solutions. Linear combinations of vectors in $\mathbb{R}^n$. The matrix equation $A\mathbf{x} = \mathbf{b}$: linearity principle and solution in parametric vector form. Spanning and linear independence in $\mathbb{R}^n$. Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$: matrix representation, one-to-one and onto transformations.

Matrix operations and properties. The inverse of a matrix: definition, properties and use. Algorithm for inverting an $n \times n$ matrix using elementary matrices: formula for $n = 2$. Characterizations of invertible matrices.

Determinants: motivation, definition and general cofactor expansions. Effect of row operations on the determinant: characterization of invertible matrices via the determinant. Determinants of transposes and products. Interpretation of the determinant as a volume.

Axioms for a vector space. Subspaces. The column and null space of a matrix; the kernel and range of a linear transformation. Spanning, linear independence and bases: general definitions. Coordinate systems with respect to a basis. The dimension of a vector space. The rank of a matrix and of a linear transformation: computation using echelon form. Rank-nullity theorem.

Eigenvalues and eigenspaces: definition, visual interpretation, and computation for 3×3 matrices. Linear independence of vectors corresponding to distinct eigenvalues. The characteristic equation. Similar matrices. Diagonalization and eigenvector bases.

Ordinary Differential Equations

Modeling via differential equations. Separation of variables. Linear first order equations: use of an integrating factor. Linear second order equations with constant coefficients.