Graduate Qualifying Exam – Solutions – Spring 2018

1. A girl is standing still, watching a bird that is flying horizontally in a straight path at a rate of 30 feet per second, at a height of 300 feet above the girl. The bird passes over the girl and continues along its path. How fast is the distance between the bird and the girl changing when the bird is 500 feet away from the girl?

   Solution. Let D be the distance between the girl and the bird. From the right triangle below, $D(x) = \sqrt{x^2 + 300^2}$, where x is the horizontal distance the bird has flown since passing over the girl. We want $\frac{dD}{dt}$. Here it is: $\frac{dD}{dt} = \frac{dD}{dx} \cdot \frac{dx}{dt} = \frac{x}{\sqrt{x^2+300^2}} \cdot 30 = \frac{30x}{\sqrt{x^2+300^2}}$.

   Finally, using the Pythagorean theorem, we see that, when $D = 500$, we have $x = 400$. This gives $\frac{dD}{dt} = \frac{30x}{\sqrt{x^2+300^2}} = \frac{12000}{500} = 24$ ft/sec.

2. Consider the region on the x-y plane that is bounded by $y = (x-1)^2$, the x-axis, and the y-axis. Find the volume generated when this region is rotated about the line $x = -1$.

   Solution. For any value of $y \in (0, 1)$, a slice through the region parallel through the x-axis, but at height y yields a disk. The outer radius of the disk is $1 + (1 - \sqrt{y}) = 2 - \sqrt{y}$, and the inner radius is 1. The area of the disk is then $\pi\left[(2-\sqrt{y})^2 - 1^2\right]$. The volume of the region is

$$
\begin{aligned}
V &= \int_0^1 \pi\left[(2-\sqrt{y})^2 - 1^2\right] dy \\
&= \int_0^1 \pi(3 - 4\sqrt{y} + y)dy = \cdots = \frac{5}{6}\pi.
\end{aligned}
$$

3. A manufacturer's production is modeled by the Cobb-Douglas function

$$f(x, y) = Kx^{\alpha}y^{1-\alpha},$$

   where K and α are constants with $K > 0$ and $0 < \alpha < 1$. In the Cobb-Douglas function, x represents the amount of labor used, and y represents the amount of capital. Each labor unit costs m dollars and each capital unit costs n dollars. Suppose the company can spend only p dollars on the production.

   (a) Find the relationship satisfied by m, n, x, y, and p that expresses the idea that the company will spend p dollars on the production.

   (b) Find the values of x and y that maximize the production function for this company, in terms of α, m, n, and p.

   Solution. (a) $mx + ny = p$.

   (b) Lagrange multipler solution: $\nabla f = (K\alpha x^{\alpha-1}y^{1-\alpha}, K(1-\alpha)x^{\alpha}y^{-\alpha})$, and, if $g = mx + ny$, then $\nabla g = (m, n)$. Setting $\nabla f = \lambda \nabla g$, yields

$$
\begin{aligned}
K\alpha x^{\alpha-1}y^{1-\alpha} &= \lambda m \qquad (1)\\
K(1-\alpha)x^{\alpha}y^{-\alpha} &= \lambda n \qquad (2)
\end{aligned}
$$

   The third equation in our system is:

$$mx + ny = p \qquad (3)$$

   This gives three equations and three unknowns (namely, x, y, and λ). There are a variety of ways to solve. In the end, you will get $x = \frac{\alpha p}{m}$, and $y = \frac{(1-\alpha)p}{n}$.

4. Consider the following double integral

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-(y-x)^2-y^2}\, dxdy.$$

(a) Express I as an integral in the new variables $u = x + y$, $v = -x + y$.

(b) Compute the exact value of I assuming as known that $\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$.

Solution. (a) $\mathbb{R}^2 \ni (x, y) \mapsto (u, v) \in \mathbb{R}^2$ is a differentiable bijection for which the Jacobian $\partial(u, v)/\partial(x, y) = \frac{1}{2}$. Moreover

$$x^2 + (y - x)^2 + y^2 = \frac{1}{4}(u - v)^2 + v^2 + \frac{1}{4}(u + v)^2 = \frac{1}{2}(u^2 + 3v^2).$$

Thus: $I = \frac{1}{2} \int\int_{\mathbb{R}^2} e^{-u^2/2-(3v^2)/2}\, dudv.$

(b) Splitting and obvious changes of variables for the resulting integrals in u, v gives

$$I = \frac{1}{2}\sqrt{2\pi}\sqrt{2\pi/3} = \pi/\sqrt{3}.$$

5. (a) Find the interval of convergence of the power series $\sum_{n\geq 1} \frac{x^n}{n}$.

(b) Find a closed form representation for $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

(c) Find the exact value of $\sum_{n=1}^{\infty} \frac{1}{n(2018)^n}$.

Solution. (a) For example, by Ratio Test the series converges absolutely for $x \in (-1, 1)$ and diverges on $\mathbb{R} \setminus [-1, 1]$. At the end points: if $x = 1$, the series diverges, while by the Alternating Series Test, at $x = -1$ the series converges. The interval of convergence is $[-1, 1)$.

(b) Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$. Term-by-term differentiation gives

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}, -1 \leq x < 1.$$

Since $f(0) = 0$, integrating now gives $f(x) = -\ln(1 - x)$.

(c) The value is $f(2018^{-1}) = \ln(2018/2017)$.

6. The sequence $(a_n)_{n\geq 1}$ is defined recursively by $a_1 > 0$ and

$$a_{n+1} = \frac{na_n}{n + a_n^2}, n \geq 1.$$

(a) Show that $(a_n)_{n\geq 1}$ is a bounded and monotonic sequence.

(b) Let $k \geq 1$. Express $\dfrac{1}{a_{k+1}} - \dfrac{1}{a_k}$ as a function of the ratio $\dfrac{a_k}{k}$.

(c) Use parts (a), (b) to show that, if $l = \lim\limits_{n\to\infty} a_n$, then for all $n \geq 1$ we have

$$\frac{1}{a_{n+1}} \geq \frac{1}{a_1} + l\sum_{k=1}^{n} \frac{1}{k},$$

(d) Compute the exact value of l.

Solution. (a) By induction $a_n > 0$ for all $n \geq 1$. Also

$$\frac{a_{n+1}}{a_n} = \frac{n}{n + a_n^2} < 1,$$

so (a_n) is strictly decreasing; in particular $a_n < a_1$, so (a_n) is bounded below by 0 and above by a_1.

(b) By the recursion, $a_{k+1}^{-1} - a_k^{-1} = a_k/k$.

(c) First of all l exists and is finite; moreover, $l \geq 0$ and $l \leq a_n$ for all $n \geq 1$. Summing over k between 1 and n all the identities in (b), we get

$$\frac{1}{a_{n+1}} = \frac{1}{a_1} + \sum_{k=1}^{n} \frac{a_k}{k} \geq \frac{1}{a_1} + l\sum_{k=1}^{n} \frac{1}{k}.$$

(d) Assuming that $l > 0$ and passing to the limit in (c), we get $1/l \geq \infty$, contradiction. Thus $l = 0$.

7. Assume that A, B are (real) matrices of dimensions $m \times n$ and $n \times p$, respectively.

(a) Show that the null space of B is a subset of the null space of AB.

(b) Prove that $\text{rank}(AB) \leq \text{rank}(B)$.

(c) State two square matrices A, B for which the inequality in (b) is strict.

Solution. (a) Let $\vec{x} \in \mathbb{R}^p$. If $\vec{x} \in \text{Nul}(B)$ then $B\vec{x} = \vec{0}$. Multiplying by A on the left of both sides gives $AB\vec{x} = \vec{0}$; that is $\vec{x} \in \text{Nul}(AB)$.

(b) By the Rank Theorem, we have

$$\dim(\text{Nul}(B)) + \text{rank}(B) = \dim(\text{Nul}(AB)) + \text{rank}(AB) = p.$$

By (a), $\dim(\text{Nul}(B)) \leq \dim(\text{Nul}(AB))$. The conclusion follows.

(c) For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = I_2$.

8. Let A be a 2×2 matrix that satisfies $A^2 + 3A - 2I_2 = O_2$; I_2 and O_2 denote the 2×2 identity, respectively zero, matrices.

   (a) Find the possible values of an eigenvalue of A.

   (b) Show that A is invertible.

   (c) Simplify the expression $B = \left(\frac{1}{2}A + A^{-1}\right)^2 - \frac{13}{4}I_2$. Then, compute the determinant of B.

   Solution. (a) Let λ be an eigenvalue of A and $\vec{v} \neq \vec{0}$ be so that $A\vec{v} = \lambda\vec{v}$. Then $A^2\vec{v} = \lambda^2\vec{v}$ and plugging all this in we get

$$(\lambda^2 + 3\lambda - 2)\vec{v} = \vec{0} \Leftrightarrow \lambda \in \{(-3 \pm \sqrt{17})/2\}.$$

   (b) We have $A(A/2 + 3I_2/2) = I_2$; so A is invertible and $A^{-1} = \frac{1}{2}A + \frac{3}{2}I_2$.

   (c) Using the information from (c), we have

$$B = \left(A + \frac{3}{2}I_2\right)^2 - \frac{13}{4}I_2 = A^2 + 3A + \frac{9}{4}I_2 - \frac{13}{4}I_2 = 2I_2 - I_2 = I_2.$$

   So $\det B = 1$.

9. A large company has total assets of \$3 trillion invested in the U.S., Asia, and Europe. At the start, none of the money is in the U.S., \$2 trillion is in Asia, and \$1 trillion is in Europe. Each year, half of the money in the U.S. stays in the U.S., $\frac{1}{4}$ goes to Asia, and $\frac{1}{4}$ goes to Europe. For each of Asia and Europe, half of the money stays home and half is sent to the U.S.

   (a) Find the matrix, A, that gives

$$\begin{bmatrix} \text{U.S.} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{[\text{year } k+1]} = A \begin{bmatrix} \text{U.S.} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{[\text{year } k]}$$

   (b) Find the eigenvalues and eigenvectors of A.

   (c) Write the vector $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ as a linear combination of the eigenvectors.

   (d) If year 0 is the initial year, with no money in the U.S., \$2 trillion in Asia, and \$1 trillion in Europe, find the distribution of the money in year k.

   Solution. (a) $\text{U.S.}_{k+1} = 0.5(\text{U.S.})_k + 0.5(\text{Asia})_k + 0.5(\text{Europe})_k$, etc., which yields

$$\begin{bmatrix} \text{U.S.} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{[\text{year } k+1]} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \text{U.S.} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{[\text{year } k]}$$

   (b)

$$det\left(\begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} - \lambda & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} - \lambda \end{bmatrix}\right) = \cdots = \left(\frac{1}{2} - \lambda\right)\lambda(\lambda - 1),$$

so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$, and $\lambda_3 = 0$.

Eigenvectors: The equation $A - \lambda_1 I = \vec{0}$ is solved by any vector of the form $[\ 2a \quad a \quad a\ ]^T$. I took $[\ 2 \quad 1 \quad 1\ ]^T$.

I find the other two eigenvectors similarly. I get $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and $\vec{x}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ as the three eigenvectors.

(c) The equation to be solved is: $\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

The solution is $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

(d) $A^k \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = A^k \left(\frac{3}{4}\vec{x}_1 + \frac{1}{2}\vec{x}_2 + \frac{3}{4}\vec{x}_2 \right) = \frac{3}{4}\lambda_1^k \vec{x}_1 + \frac{1}{2}\lambda_2^k \vec{x}_2 + \frac{3}{4}\lambda_3^k \vec{x}_2 = \cdots = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{4} + \left(\frac{1}{2}\right)^{k+1} \\ \frac{3}{4} - \left(\frac{1}{2}\right)^{k+1} \end{bmatrix}$.

Alternate Solution for part (d): A can be diagonalized as $A = SDS^{-1}$, with $S = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$. S^{-1} is found to be $\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$. Then $\begin{bmatrix} \text{U.S.} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{[\text{year } k]} = A^k \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1}{2}\right)^k & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{4} + \left(\frac{1}{2}\right)^{k+1} \\ \frac{3}{4} - \left(\frac{1}{2}\right)^{k+1} \end{bmatrix}$.

10. (a) Find the general solution $y(t)$ of the differential equation $5y'' + 10y' + 6y = 0$.

(b) Compute $\lim_{t \to \infty} y(t)$.

(c) Assuming that $y(0) \neq 0$, prove that there is a value $t_0 \in \mathbb{R}$ for which $|y(t_0)| > 1$.

(d) Use parts (b) and (c) to show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(t) = \dfrac{y^2(t)}{1 + y^4(t)}$ attains a maximum value of 0.5.

Solution. (a) The characteristic equation is $5\lambda^2 + 10\lambda + 6 = 0$ with roots $-1 \pm i/\sqrt{5}$. Thus, the general solution is

$$y(t) = c_1 e^{-t} \cos(t/\sqrt{5}) + c_2 e^{-t} \sin(t/\sqrt{5}), c_1, c_2 \in \mathbb{R}.$$

(b) Clearly, the limit is 0. Enough to see that $|y(t)|$ is dominated by e^{-t}.

(c) $y(0) = c_1 \neq 0$. Now, pick a sufficiently large integer k so that $|c_1| e^{\sqrt{5}k\pi} > 1$ and let $t_0 := -\sqrt{5}k\pi$.

(d) Clearly $0 \leq f(t) \leq 1/2$. The maximum value 1/2 is attained if there is some t_1 such that $|y(t_1)| = 1$. But based on (b), (c) this is guaranteed by the Intermediate Value Theorem.