Graduate Qualifying Exam – Fall 2018 – Solutions

Problem 1. A rectangular box with its base in the xy-plane is inscribed under the graph of the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$. Find the maximum possible volume of the box, and rigorously justify that you have found the maximum.

By symmetry, we assume the sides of the box are parallel to the vertical coordinate planes. In addition, we may assume that if the base corner in the first quadrant is (x, y), then the other corners are at $(-x, y)$, $(-x, -y)$ and $(x, -y)$. The total volume of the box is thus $V(x, y) = 4xy(1-x^2-y^2)$. Since we are assuming $x, y \geq 0$, we note that if either is greater than one, the box is no longer under the paraboloid, so it suffices to maximize $V(x, y)$ subject to the assumption that $0 \leq x \leq 1$ and $0 \leq y \leq 1$. On the boundary of the region we get values of V that are less than or equal to zero, so we check interior points for which $\partial V/\partial x = 0 = \partial V/\partial y$. We find that $4y(1 - 3x^2 - y^2) = 0$ and $4x(1 - x^2 - 3y^2) = 0$. Since $V(0, 0) = 0$, we may assume x and y are nonzero, which allows us to deduce that $y = x$ or $y = -x$. Since the latter equation only has $(0, 0)$ as a solution, our optimum values are those for which $y = x$. Plugging this into $1 = 3x^2 + y^2$ yields $x = 1/2$ and pluggin this into our volume function yields $V(1/2, 1/2) = 1/2$. Notice that this must be a maximum as it is our only extreme point in the closed bounded region.

Problem 2. Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 2x$.

We use cylindrical coordinates. The base region on the $r\theta$-plane over which we integrate is bounded by the ellipse $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$, so our integral is

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_{r^2}^{2r\cos\theta} r dz dr d\theta.$$

We have

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_{r^2}^{2r\cos\theta} r dz dr d\theta &= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (2r^2\cos\theta - r^3) dr d\theta \\
&= \int_{-\pi/2}^{\pi/2} (\frac{2}{3}(2\cos\theta)^3 \cos\theta - \frac{1}{4}(2\cos\theta)^4) d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{4}{3} \cos^4\theta d\theta \\
&= \frac{4}{3} \int_{-\pi/2}^{\pi/2} \frac{1}{4}(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}) \\
&= \pi/2.
\end{aligned}$$

Problem 3.

(a) Prove that $\ln(1+x^{-1}) > \frac{1}{1+x}$ for $x > 0$.

By rearranging and taking exponentials, it suffices to prove that $(1+x^{-1})e^{-\frac{1}{1+x}} > 1$ for $x > 0$. To establish this inequality, we expand the exponential in a power series so that the first two terms of $(1+x^{-1})e^{-\frac{1}{1+x}}$ are $\frac{x+1}{x} + -\frac{1}{x} = 1$ while, for n even, the n plus $n+1$ term is

$$\frac{1}{xn}(-1)^n\left(\frac{1}{1+x}\right)^{n-1}\left(1-\frac{1}{n(1+x)}\right)$$

which is greater than zero.

(b) Prove that $x\ln(1+x^{-1})$ is strictly increasing for $x > 0$.

Taking derivatives we obtain $\ln(1+x^{-1}) - \frac{1}{x+1}$ so the result now follows from part (a).

(c) Compute $\lim(x\ln(1+x^{-1}))$ as $x \to 0$ and as $x \to \infty$.

The first limit equals $\lim\limits_{x\to 0} \frac{\ln(1+x^{-1})}{x^{-1}}$ so L'Hopitals rule implies it is $\lim\limits_{x\to 0} \frac{1}{(1+x^{-1})} = 0$. Similarly, the other limit is 1.

Problem 4.

(a) Find all points on the ellipsoid $2x^2 + 3y^2 + z^2 = 9$ whose tangent planes are parallel to the yz-plane.

Let $G(z, y, z) = 2x^2+3y^2+z^2-9$. Then for a point Q on the ellipsoid, $\nabla G(Q)$ is perpendicular to the tangent plane of the ellipsoid at Q. Thus, we need all $Q = (x, y, z)$ on the ellipsoid such that the vector $4x\vec{i} + 6y\vec{j} + 2z\vec{k}$ has zero y and z component. We find two points, $(\pm\frac{3}{\sqrt{2}}, 0, 0)$.

(b) The point $P = (1, -1, 2)$ lies on both the paraboloid $x^2+y^2 = z$ and the ellipsoid $2x^2+3y^2+z^2 = 9$. Write an equation of the plane through P that is normal to the curve of intersection of these two surfaces.

Let $F(x, y, z) = x^2+y^2-z$. Then $\nabla F(P)$ is normal to the tangent plane of the paraboloid at P and $\nabla G(P)$ is normal to the tangent plane of the ellipsoid at P. Therefore, $\vec{n} = \nabla F(P)\times\nabla G(P)$ is tangent to the intersection curve at P, so we need the equation of a plane through P with normal vector $\vec{n}$. We find $\nabla F(P) = 2\vec{i} - 2\vec{j} - \vec{k}$ while $\nabla G(P) = 4\vec{i} - 6\vec{j} + 4\vec{k}$, so that $\vec{n} = -14\vec{i} - 12\vec{j} - 4\vec{k}$, which is proportional to $7\vec{i} + 6\vec{j} + 2\vec{k}$. This gives the plane $7x + 6y + 2z = 5$.

Problem 5. Let $a, b, c \in \mathbb{R}$ and suppose a, b and c are not all zero. Find the general solution to the differential equation $ay'' + by' + cy = 0$ according to the values of a, b and c.

The characteristic equation for this differential equation is $ar^2 + br + c = 0$. There are several cases. If $a = 0$ and $b = 0$, then y is constant. If $a = 0$ and $b \neq 0$, then $y = \alpha e^{-\frac{cx}{b}}$ where α is a constant. If $a \neq 0$ and $b^2 - 4ac > 0$, then y has the form $c_1e^{\alpha x} + c_2e^{\beta x}$ where α and β are the distinct roots of the characteristic equation. If $b^2 = 4ac$, then y has the form $(c_1 + c_2x)e^{\alpha x}$ where α is the root of the characteristic equation. Finally, if $b^2 - 4ac < 0$, then there is a pair of complex roots of the form $\alpha \pm \beta i$, and y has the form $(c_1\cos\beta x + c_2\sin\beta x)e^{\alpha x}$.

Problem 6. Let V and W be vector spaces, and suppose that $T : V \to W$ is a linear transformation.

(a) Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ be vectors in V. If the set $\{T(\mathbf{v}_1), ..., T(\mathbf{v}_p)\}$ is linearly independent, what conclusion, if any, can you draw about the linear independence of the set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$?

We will show that the set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ is also linearly independent. Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ for some scalars $c_1, ..., c_p$. Applying T and using the properties of linear transformations yields $c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_pT(\mathbf{v}_p) = T(\mathbf{0}) = \mathbf{0}$. Since $\{T(\mathbf{v}_1), ..., T(\mathbf{v}_p)\}$ is linearly independent, then $c_1 = c_2 = \cdots = c_p = 0$, hence $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ is linearly independent.

(b) Show that the kernel of T, $\{v \in V : T(v) = \mathbf{0}\}$, is a subspace of V.

Let $K = \{v \in V : T(v) = \mathbf{0}\}$. Clearly $K \subseteq V$. Since $T(\mathbf{0}) = \mathbf{0}$, we have $\mathbf{0} \in K$. If $\mathbf{u}, \mathbf{v} \in K$, then $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Then if c is a scalar, $T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0}$. Thus K is nonempty and closed under vector addition and scalar multiplication, therefore K is a subspace of V.

(c) Let $T : M_{2\times 2} \to \mathbb{R}$ be the linear transformation defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$. Find a basis for the kernel of T and compute its dimension.

The kernel of T consists of all matrices of the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$. A basis for the kernel is given by $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$, so the kernel has dimension 3.

Problem 7.

(a) Let $A = \begin{bmatrix} .4 & -.3 \\ .4 & 1.2 \end{bmatrix}$. Compute $\lim_{k\to\infty} A^k$, if the limit exists, or explain why it does not.

The eigenvalues of A are 1 and 0.6. We can diagonalize A as

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.6 \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}.$$

Then for a positive integer k, we have

$$A^k = \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (0.6)^k \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix} \to \frac{1}{4} \begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix}$$

as $k \to \infty$.

(b) Suppose an $n \times n$ matrix A has n distinct eigenvalues, each less than 1 in absolute value. Find $\lim_{k\to\infty} A^k$ and justify your answer.

Since A has n distinct eigenvalues, it is diagonalizable, so $A = PDP^{-1}$ for some invertible matrix P and diagonal matrix D. Then for any positive integer k, $A^k = (PDP^{-1})^k = PD^kP^{-1}$. Then D^k is the diagonal matrix whose entries are the corresponding entries of D raised to the kth power. Since the entries of D are the eigenvalues of A, they are each less than one in absolute value, so as $k \to \infty$, D^k approaches the zero matrix. Therefore $A^k \to 0$ as $k \to \infty$.

Problem 8. A television camera is positioned on the ground 4000 feet from the base of a rocket launching pad. The camera rotates to keep the rocket in view, and the automatic focusing mechanism must take into account the changing distance between the camera and the rocket. Suppose a rocket launched from the pad rises vertically at 600 ft/s when the rocket is at 3000 feet.

(a) How fast is this distance between the camera and the rocket changing at that moment?

The camera, launch pad, and rocket form three vertices of a right triangle, whose hypotenuse, of length z, is the line between the camera and the rocket. Let y be the length of the side representing the line between the rocket and the launch pad. Then y and z are related by $y^2 + 16000 = z^2$. Differentiating with respect to time t, we have $2y\frac{dy}{dt} = 2z\frac{dz}{dt}$. When $y = 3000$, $z = 5000$ and $\frac{dy}{dt} = 600$ ft/s, so the distance z is changing at the rate of $\frac{dz}{dt} = 360$ ft/s.

(b) How fast is the angle between the rocket and the ground, from the point of view of the camera, changing at that moment?

Let θ be the angle between the camera's line of sight and the ground. Then we have $\cos\theta = \frac{4000}{z}$. Differentiating with respect to time t gives $-\sin\theta\frac{d\theta}{dt} = -\frac{4000}{z^2}\frac{dz}{dt}$. Using part (a) and the fact that $\sin\theta = \frac{3}{5}$ at the moment in question, we have $\frac{d\theta}{dt} = 0.096$ radians per second.

Problem 9. Determine whether each series below converges or diverges. If a series converges, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$

This series diverges by the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$, which is a divergent p-series.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$

This series converges by the ratio test. Alternatively, we can both show the series converges and find its sum by considering its sequence of partial sums. Let s_n be the nth partial sum of the series. We can show by induction that $s_n = \frac{(n+1)!-1}{(n+1)!} = 1 - \frac{1}{(n+1)!}$. Then as $n \to \infty$, $s_n \to 1$. Thus the sum of the series is 1.

Problem 10. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 - 2, & \text{if } x \in \mathbb{Q}, \\ 2 - 3x, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Determine the points of continuity of f. Choose one such point and give a careful proof of the continuity of f at that point using the $\epsilon - \delta$ definition. Completely justify the fact that f is discontinuous at all other points.

The function f is continuous only at $x = 1$ and $x = -4$. We will show continuity at $x = 1$. The proof for $x = -4$ is similar. Let $\epsilon > 0$ and choose $\delta = \min\{1, \epsilon/3\}$. Then for all x such that $|x - 1| < \delta$, we have $0 < x < 2$, so $1 < x + 1 < 3$. Further, $|f(x) - f(1)| = |f(x) + 1|$. We have two cases. If $x \in \mathbb{Q}$, then $|f(x) + 1| = |x^2 - 1| = |x + 1||x - 1| < 3\delta \leq \epsilon$. If $x \notin \mathbb{Q}$, then we have $|f(x) + 1| = |3 - 3x| = 3|x - 1| < 3\delta \leq \epsilon$. Hence f is continuous at $x = 1$. To see that f is discontinuous at every other point, note that if $c \neq 1, -4$, then $c^2 - 2 \neq 2 - 3c$. Since every interval of real numbers contains both rational and irrational numbers, then every δ-interval about c must contain points at a distance at least $|c^2 - 2 - (2 - 3c)|$ from each other.