

Graduate Qualifying Exam – Fall 2018

Directions: In all problems you must show your work in order to receive credit. You may not use a calculator. All electronic devices must be turned off. You have three hours.

Problem 1. A rectangular box with its base in the xy -plane is inscribed under the graph of the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$. Find the maximum possible volume of the box, and rigorously justify that you have found the maximum.

Problem 2. Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 2x$.

Problem 3.

- (a) Prove that $\ln(1 + x^{-1}) > \frac{1}{1+x}$ for $x > 0$.
- (b) Prove that $x \ln(1 + x^{-1})$ is strictly increasing for $x > 0$.
- (c) Compute $\lim(x \ln(1 + x^{-1}))$ as $x \rightarrow 0$ and as $x \rightarrow \infty$.

Problem 4.

- (a) Find all points on the ellipsoid $2x^2 + 3y^2 + z^2 = 9$ whose tangent planes are parallel to the yz -plane.
- (b) The point $P = (1, -1, 2)$ lies on both the paraboloid $x^2 + y^2 = z$ and the ellipsoid $2x^2 + 3y^2 + z^2 = 9$. Write an equation of the plane through P that is normal to the curve of intersection of these two surfaces.

Problem 5. Let $a, b, c \in \mathbb{R}$ and suppose a, b and c are not all zero. Find the general solution to the differential equation $ay'' + by' + cy = 0$ according to the values of a, b and c .

Problem 6. Let V and W be vector spaces, and suppose that $T : V \rightarrow W$ is a linear transformation.

- (a) Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ be vectors in V . If the set $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly independent, what conclusion, if any, can you draw about the linear independence of the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$?
- (b) Show that the kernel of T , $\{v \in V : T(v) = \mathbf{0}\}$, is a subspace of V .
- (c) Let $T : M_{2 \times 2} \rightarrow \mathbb{R}$ be the linear transformation defined by $T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + d$. Find a basis for the kernel of T and compute its dimension.

Problem 7.

- (a) Let $A = \begin{bmatrix} .4 & -.3 \\ .4 & 1.2 \end{bmatrix}$. Compute $\lim_{k \rightarrow \infty} A^k$, if the limit exists, or explain why it does not.
- (b) Suppose an $n \times n$ matrix A has n distinct eigenvalues, each less than 1 in absolute value. Find $\lim_{k \rightarrow \infty} A^k$ and justify your answer.

Problem 8. A television camera is positioned on the ground 4000 feet from the base of a rocket launching pad. The camera rotates to keep the rocket in view, and the automatic focusing mechanism must take into account the changing distance between the camera and the rocket. Suppose a rocket launched from the pad rises vertically at 600 ft/s when the rocket is at 3000 feet.

- (a) How fast is this distance between the camera and the rocket changing at that moment?
- (b) How fast is the angle between the rocket and the ground, from the point of view of the camera, changing at that moment?

Problem 9. Determine whether each series below converges or diverges. If a series converges, find its sum.

(a) $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$

(b) $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$

Problem 10. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 - 2, & \text{if } x \in \mathbb{Q}, \\ 2 - 3x, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Determine the points of continuity of f . Choose one such point and give a careful proof of the continuity of f at that point using the $\epsilon - \delta$ definition. Completely justify the fact that f is discontinuous at all other points.