Qualifying Exam

Spring 2019
April 1, 2019

Number ____________

You may use calculators for this exam. — Justify all your answers. Answer specific questions by giving the exact values, not approximations.

Problem 1. Consider the unit sphere. Determine the radius of a circular cylinder whose axis is along the sphere's diameter and which contains one-half of the sphere's volume. See Figure 1.

Problem 2. Find all the solutions of the equation

$$y'(x) = |y(x)|, \quad x \in \mathbb{R}$$

which are defined for all $x \in \mathbb{R}$.

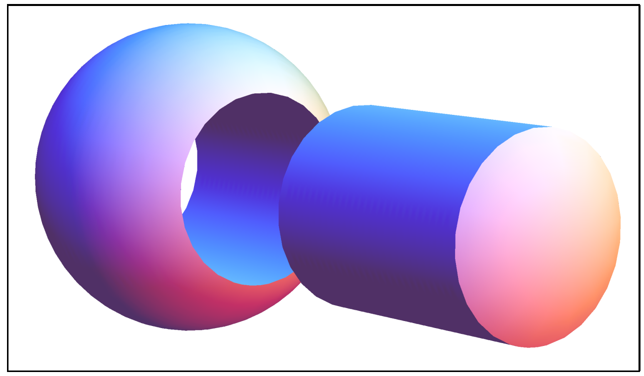

Fig. 1: The unit sphere cut by a cylinder

Problem 3. Figure 2 shows the parabola $y = x^2$ and a unit circle with its center on the y-axes. This unit circle intersects the parabola at the right angle. That means that at each point of intersection the tangent lines to the parabola and the unit circle are perpendicular. Find the center of this unit circle.

Problem 4. Let P be a fixed point on the unit circle $\mathbb{T}$. Denote by a_e the average of the Euclidian distances of all the points on $\mathbb{T}$ to the point P. (Two Euclidean distances are indicated as blue line segments in Figure 3.) Denote by a_l the average of all shortest arc lengths from all the points on $\mathbb{T}$ to P. (Two shortest arc lengths are shown as teal circular arcs in Figure 3.)

(a) Use common sense about the averages to order the numbers 0, 1, 2, a_e, a_l and π in the increasing order. Explain your reasoning in as simple terms as possible without doing any calculus.

(b) Calculate a_l exactly using integrals.

(c) Calculate a_e exactly using integrals.

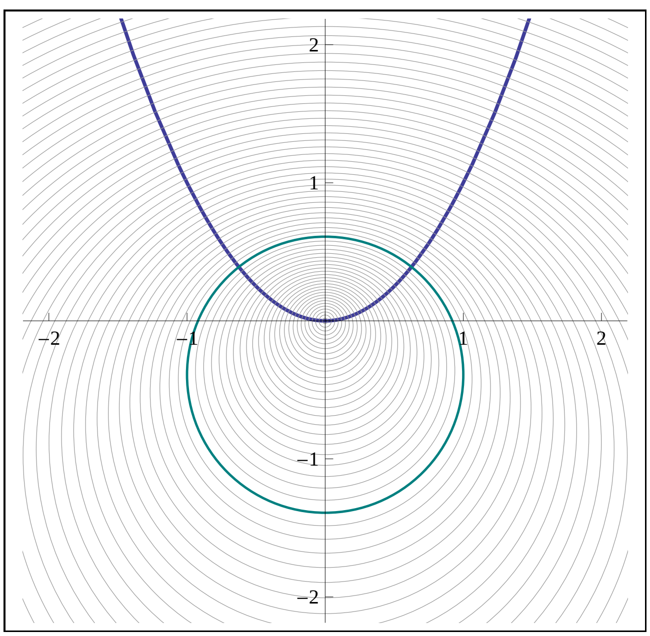

Fig. 2: $y = x^2$ and a unit circle

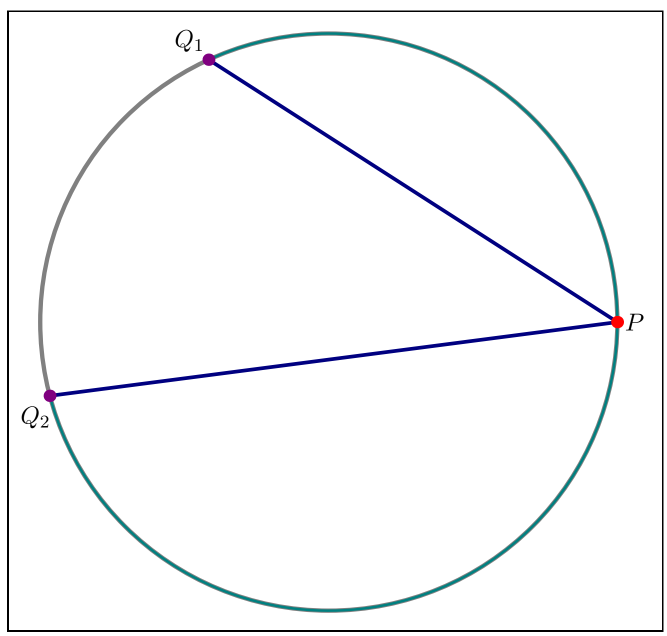

Fig. 3: The unit circle and averages

Problem 5. (a) Find the Maclaurin series for $(\sin x)^2$ using the Maclaurin series for $\cos(2x)$.

(b) Using the answer for part (a), find $\lim_{x\to 0} \dfrac{(\sin x)^2 - x^2}{x^4}$.

Problem 6. Let $t \in \mathbb{R}$ and consider the functions $f_t : \mathbb{R} \to \mathbb{R}$ given by

$$f_t(x) = x + te^{-x^2} \quad \text{for all} \quad x \in \mathbb{R}.$$

(a) Prove that for each $t \in \mathbb{R}$ the function f_t is a surjection.

(b) Prove that the function f_t is a bijection if and only if $|t| \leq \sqrt{e/2}$.

Problem 7. Let A be a real 3×2 matrix and let B be a real 2×3 matrix. Prove the following implication:

$$AB = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & -4 & 3 \end{bmatrix} \qquad \Rightarrow \qquad BA = I_2$$

Hint: AB is singular, $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ is an eigenvector of AB and AB has an eigenvalue of multiplicity 2.

Problem 8. Given

$$A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 2 & 2 & -3 \\ 2 & 3 & 3 & -5 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

find a vector $\mathbf{v} \in \operatorname{Nul} A$ and a vector $\mathbf{w} \in \operatorname{Row} A$ such that $\mathbf{y} = \mathbf{v} + \mathbf{w}$.
Hint that you need not use: Such vectors $\mathbf{v}$ and $\mathbf{w}$ are uniquely determined.

Problem 9. A classical calculus problem is as follows: Consider a piece of wire of length 4 and cut it in two pieces. Make one piece into a square and the other piece into a circle. Find the minimum and the maximum of the total area that is enclosed by such formed square and circle.
In this problem we ask you to solve the analogous problem for a cube and a sphere: Assume that the total surface area of a cube and a sphere is 4. Calculate the minimum and the maximum of the total volume enclosed by such a cube and a sphere.

Problem 10. Let $f : \mathbb{N} \to \mathbb{N}$ be a function where $\mathbb{N}$ denotes the set of positive integers.

(a) Prove that $\displaystyle\sum_{n=1}^{+\infty} \frac{f(n)}{n^2}$ diverges whenever f is strictly increasing.

(b) Prove that there exists nondecreasing unbounded f such that $\displaystyle\sum_{n=1}^{+\infty} \frac{f(n)}{n^2}$ converges.