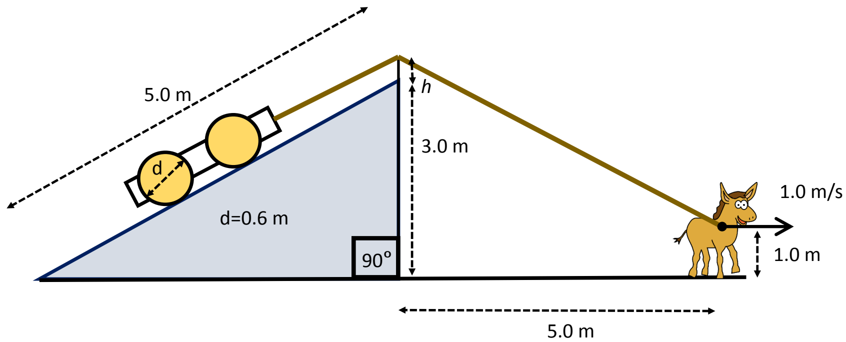
Qualifying Exam – Spring 2017

March 27, 2017

- The allocated time for this exam is 180 minutes. All problems have equal weight.
- **Calculators are permitted. Show your work for full credit.**



1. A donkey pulls a cart up a ramp of length 5.0 m and height 3.0 m. The rope is attached to a pulley on a pole of height h at the top of the ramp. The rope is 10 m long. When pulled taut, it is parallel to the ramp. The diameter of the wheels of the cart is 0.6 m. The rope is attached to the shoulders of the donkey at height 1.0 m.

   (a) What is the height h of the pole on top of the ramp?

   (b) What is the angular velocity of the wheels (in radians per second) when the donkey is 5.0 m from the foot of the ramp and moving at a speed of 1.0 m/s?

2. Assume that $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$. Let $s = \sum_{n=1}^{\infty} a_n$ and $t = \sum_{k=0}^{\infty} 2^k a_{2^k}$.

   (a) Verify that s converges if and only if t converges.

   (b) Show that the monotonicity assumption cannot be dropped:

      i. Find $b_1, b_2, \ldots \geq 0$ such that $\sum_{k=0}^{\infty} 2^k b_{2^k}$ converges, but $\sum_{n=1}^{\infty} b_n$ diverges.

      ii. Find $c_1, c_2, \ldots \geq 0$ such that $\sum_{n=1}^{\infty} c_n$ converges, but $\sum_{k=0}^{\infty} 2^k c_{2^k}$ diverges.

3. Suppose the function $f(x) = \sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $0 < \rho < \infty$.

   (a) What is the radius of convergence of $g(x) = \sum_{n=1}^{\infty} a_n^2 x^n$? Prove your statement.

   (b) Prove or disprove: If $f(x)$ converges at $x = 1$, then $g(x)$ also converges at $x = 1$.

   (c) Prove or disprove: If $f(x)$ diverges at $x = 1$, then $g(x)$ also diverges at $x = 1$.

   (d) What is the radius of convergence of $h(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$? Prove your statement.

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4. Consider a hill whose height is described by $h(x, y) = 200 - \dfrac{x^2}{200} - \dfrac{y^2}{200}$, measured in meters.

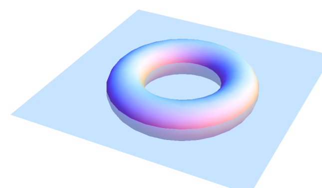
   (a) At $t = 0$, you start walking on the hill at position $(50, 50, 175)$. Your horizontal speed is 2 m/s and you set out in the direction $\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$. At what rate is your altitude changing at $t = 0$?

   (b) Find the upward-pointing normal vector $\overrightarrow{n}$ of unit length to the surface of the hill at $(50, 50, 175)$.

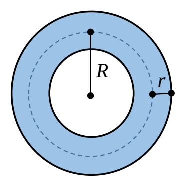
   (c) Fir trees grow vertically (in the z-direction) on the surface of the hill. Where on the hill is the angle α between the tree trunks and the normal vector given by $\alpha = \frac{\pi}{3}$?

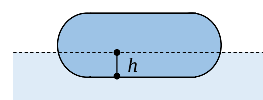
5. An inflatable water tube has the shape of a torus. The torus is generated by revolving a circle of radius r about an axis coplanar with the circle, having distance R from the center of the circle (with $r < R$). (See sketch.) The water tube is submerged to a depth of h such that the surface of the water is parallel to the plane of the large circle.

   (a) Set up an integral for the submerged volume $V(h)$ as a function of depth h. (You don't have to evaluate the integral.)

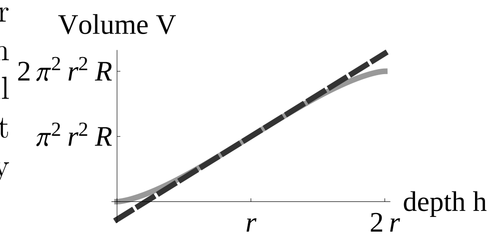


   (b) When plotting $V(h)$ versus h, one obtains the graph to the right. The dashed line is the tangent to the graph at the point $h = r$, $V = \pi^2 R r^2$. Find a formula for the tangent.





   (c) Now suppose we have an inner tube with radii $R = 75$ cm and $r = 20$ cm. Two large people weighing together 250 kg use it as a floating device. (Neither of them is touching the water.) Use the approximation in part (b) to estimate the depth of the tube. (Recall Archimedes' principle: For a floating body, its weight equals the weight of the water it displaces. The density of water is approximately 0.001 kg/cm^3.)



6. Consider a closed system with 100,000 individuals. A flu epidemic starts. Let $I(t)$ denote the proportion of infected individuals in the closed system at time t (in days). Assume that the rate at which the proportion of infected individuals is changing is jointly proportional to the proportion of infected individuals and the proportion of uninfected individuals. Furthermore, suppose that when 10% of individuals are infected, the rate at which individuals are becoming infected is 900 individuals per day. If 200 individuals are infected at time $t = 0$, when will 90% of the population be infected? (Notes: (1) The assumption here is that there are only healthy and sick individuals. Sick individuals never recover. (2) You may assume that the number of individuals is large enough that you can model the proportion of infected individuals with a differential equation.)

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7. Let $f(x,y) = -x^2 + y$ and $g(x,y) = x^2 - 2xy + y^2 - 16$.

   (a) Sketch the contours of $f(x,y)$ and the set $g(x,y) = 0$ in the xy-plane. Mark the approximate locations of the global minimizer and maximizer (if they exist) of $f(x,y)$ subject to the constraint $g(x,y) = 0$. If any of them does not exist, give a reason.

   (b) Compute the exact locations of the global minimizer and maximizer you marked above.

8. Evaluate the integral by properly reversing the order

$$\int_0^1 \int_x^1 \int_y^1 e^{z^3} \, dz \, dy \, dx.$$

9. For any natural number $n = 1, 2, \ldots$, consider the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & 2 & \cdots & 2 \\ 3 & 3 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ n & n & n & \cdots & n \end{pmatrix}.$$

   (a) Determine the dimension of the kernel of A.

   (b) Is A diagonalizable? If so, diagonalize A, i.e. find a diagonal matrix D and an invertible matrix S such that $A = SDS^{-1}$. Otherwise, give a reason.

   (c) Let $\mathbf{I}$ be the $n \times n$ identity matrix. Compute the determinant of the matrix

$$A + \mathbf{I} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 2 & 3 & 2 & \cdots & 2 \\ 3 & 3 & 4 & \cdots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ n & n & n & \cdots & n+1 \end{pmatrix}.$$

10. When a shark detects blood in the water, it will swim in the direction in which the concentration of blood increases most rapidly. Suppose that the concentration of blood at point (x,y) on the surface of seawater is given by $C(x,y) = e^{-(x^2+2y^2)/10^4}$, where x and y are measured in meters in a rectangular coordinate system with the blood source at the origin. The concentration is assumed not to change during the time period under consideration.

    (a) What is the slope of the tangent to the shark's path as it passes through a point (x,y)?

    (b) Suppose the shark starts at the point $(100, 100)$. Its path after it has swum to the origin is given by $y = f(x)$, for some function f. Determine f by properly setting up a differential equation.

END OF EXAM