Solutions for the Qualifying Exam – Spring 2017

1. (a) By geometry,
$$\frac{d/2}{h} = \frac{\sqrt{5^2 - 3^2}}{5} \quad \Rightarrow \quad h = 0.375 \text{ (m)}.$$

   (b) Set the l-axis along the ramp with the origin at the left end on the floor and the s-axis horizontal with the origin at the foot of the ramp on the floor. Suppose the left end of the rope located above the ramp is L m in terms of the l coordinate and the right end on the shoulder of the donkey is S m in terms of the s coordinate. Then
$$5 - L + \frac{3}{5}h + \sqrt{S^2 + (h + 3 - 1)^2} = 10.$$
   Differentiating with respect to time t on both sides yields
$$-\frac{dL}{dt} + \frac{S\frac{dS}{dt}}{\sqrt{S^2 + (h+2)^2}} = 0.$$
   When $S = 5.0$ m and $\frac{dS}{dt} = 1.0$ m/s, $\frac{dL}{dt} = 0.9033$ m/s. Hence the angular velocity of the wheels is
$$\frac{dL/dt}{d/2} = 3.0109 \text{ rad/s}.$$

2. (a) Let $f(x)$ be a continuous, decreasing function such that $a_n = f(n)$ for $n = 1, 2, \ldots$ By the integral test: s converges $\Leftrightarrow \int_1^\infty f(x)dx < \infty \Leftrightarrow \ln 2 \int_0^\infty 2^u f(2^u)du < \infty \Leftrightarrow t$ converges. In the next to last step, we used the substitution $x = 2^u$.

   (b) i. Choose $b_n = \begin{cases} 0 & \text{if } n = 2^k \text{ for some } k \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$

   ii. Choose $c_n = \begin{cases} 1/n & \text{if } n = 2^k \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$

3. (a) The radius of convergence of $g(x)$ is ρ^2 because
$$\limsup_{n\to\infty} \sqrt[n]{a_n^2} = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^2 = \frac{1}{\rho^2}.$$

   (b) No. $f(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges but $g(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

   (c) No. $f(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges but $g(1) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

   (d) The radius of convergence of $p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ is ∞ because for any x,
$$\limsup_{n\to\infty} \frac{|x^{n+1}|/(n+1)!}{|x^n|/n!} = \limsup_{n\to\infty} \frac{|x|}{n+1} = 0.$$

(This step is unnecessary if $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$ can be used directly.)

The radius of convergence of $h(x)$ is ∞ since

$$0 \le \limsup_{n\to\infty} \frac{\sqrt[n]{|a_n|}}{\sqrt[n]{n!}} \le \lim_{n\to\infty} \frac{1}{\rho\sqrt[n]{n!}} = 0.$$

4. (a) Let $\vec{v} = 2 \cdot \langle 1/2, \sqrt{3}/2 \rangle$. Then the rate of change of altitude is $\frac{\partial h}{\partial \vec{v}} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\sqrt{3} \approx -1.366$ meters/sec.

   (b) The equation of the hill is given by the implicit equation $f(x, y, z) = z + x^2/200 + y^2/200 = 200$, so a vector perpendicular to it is the gradient $\nabla f = (x/100, y/100, 1)^T$. Normalizing yields $\vec{n} = \frac{1}{\sqrt{x^2+y^2+10{,}000}}(x, y, 100)^T$. At $x = y = 50$, we have $\vec{n} \approx (0.408, 0.408, 0.816)^T$.

   (c) The condition is $\vec{n} \cdot (0, 0, 1)^T = \cos(\pi/3) = 1/2$. This is equivalent to $h(x, y) = 50$, so the trees have this angle at height 50 meters, or equivalently on a circle of radius 173 meters from the top of the hill.

5. (a) The area of the h-level slice of the tube is

$$S(h) = \pi\left(r_1^2(h) - r_2^2(h)\right)$$

   where

$$r_1(h) = R + \sqrt{r^2 - (r-h)^2},$$
$$r_2(h) = R - \sqrt{r^2 - (r-h)^2}.$$

   Simplify to get

$$S(h) = 4\pi R\sqrt{r^2 - (r-h)^2} = 4\pi R\sqrt{h(2r-h)}.$$

   Therefore the volume of submerged water tube is

$$V(h) = \int_0^h S(x)\,dx = 4\pi R \int_0^h \sqrt{x(2r-x)}\,dx.$$

   (b) The equation of the tangent through $(r, \pi^2 R r^2)$ is

$$V = \pi^2 R r^2 + V'(r)(h-r) = \pi^2 R r^2 + 4\pi R r(h-r).$$

   (c) Solve the following equation for h

$$250/0.001 = \pi^2 \cdot 75 \cdot 20^2 + 4\pi \cdot 75 \cdot 20(h-20) \quad \Rightarrow \quad h = 17.5549 \text{ (cm)}.$$

6. Based on the given conditions, $I(t)$ satisfies the ODE

$$\frac{dI}{dt} = kI(1-I)$$

where k is the proportionality constant to be determined. The following extra conditions are also given

   (a) When $I = 0.1$, $\dfrac{dI}{dt} = \dfrac{900}{100{,}000} = 9 \times 10^{-3}$.

(b) The initial condition $I(0) = \dfrac{200}{100,000} = 2 \times 10^{-3}$.

The first condition implies

$$9 \times 10^{-3} = k \cdot 0.1 \cdot 0.9 \quad \Rightarrow \quad k = 0.1.$$

Using separation of variables, the general solution to the ODE can be determined as

$$I(t) = \frac{ce^{kt}}{ce^{kt} + 1} = \frac{ce^{0.1t}}{ce^{0.1t} + 1}$$

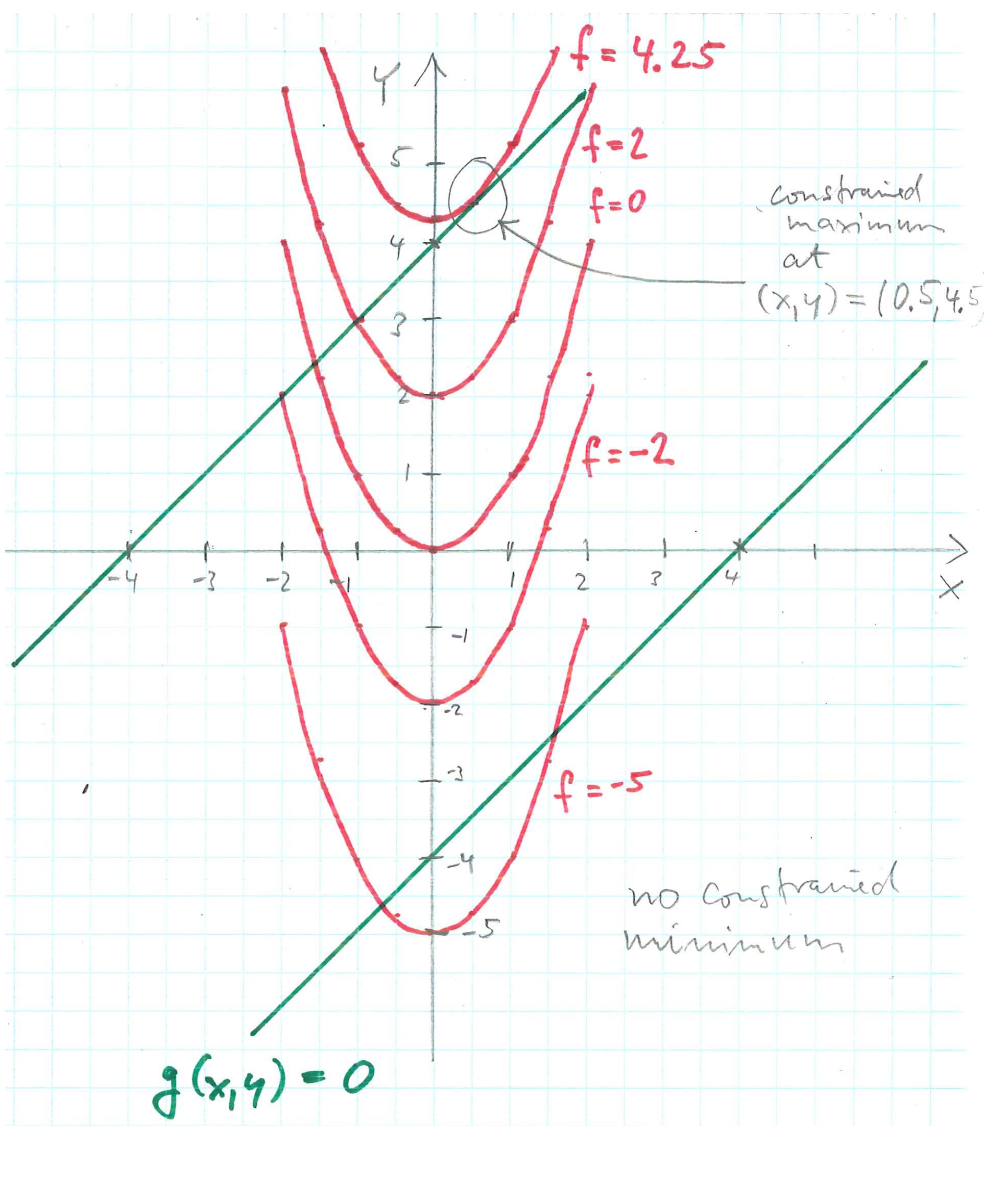
where c is an arbitrary constant.
Due to the initial condition,

$$I(0) = \frac{c}{c+1} = 2 \times 10^{-3} \quad \Rightarrow \quad c = \frac{1}{499}.$$

Solving

$$I(t) = \frac{e^{0.1t}}{e^{0.1t} + 499} = 0.9 \quad \Rightarrow \quad e^{0.1t} = 499 \cdot 9 \quad \Rightarrow \quad t = 10 \ln(499 \cdot 9) = 84.1.$$

7. (a)



(b) Using Lagrangian multipliers: $\nabla f = \lambda \nabla g \Leftrightarrow -2x = 2\lambda(x-y), 1 = -2\lambda(x-y)$ and also $x - y = -4$ (from sketch). This yields $(x,y) = (0.5, 4.5)$ as the location for the maximizer.

8. The region of integration is $\{(x,y,z)\colon 0 \le x \le y \le z \le 1\}$. Reversing the order of integration gives $\int_0^1 \int_x^1 \int_y^1 \exp(z^3)\, dz\, dy\, dx = \int_0^1 \int_0^z \int_0^y \exp(z^3)\, dx\, dy\, dz = \int_0^1 0.5z^2 \exp(z^3)\, dz = (e-1)/6$.

9. (a) Since dim(Col A)= 1, dim(Nul A)= $n-1$.

   (b) A is diagonalizable.

   Since dim(Nul A)= $n-1$, A has $n-1$ zero eigenvalues with any associated eigenvector $[v_1\ v_2\ \cdots\ v_n]^T$ satisfying

$$v_1 + v_2 + \cdots + v_n = 0.$$

   Let e_{ij} be the n-dimensional vector with the ith component being 1, the jth component being -1 and all other components being 0. Then $\{e_{k,k+1}\}$, $k = 1, 2, \cdots, n-1$, form a basis of the eigenspace of the zero eigenvalue. The only non-trivial eigenvalue must be the trace of A, which is $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ with the associated eigenvector $V = [1\ 2\ \cdots\ n]^T$. Let $S = [e_{12}\ \ e_{23}\ \ \cdots\ e_{n,n+1}\ \ V]$ and D be the diagonal matrix with the only non-zero entry located on the nth row and column being $\frac{n(n+1)}{2}$. Then $A = SDS^{-1}$.

   (c) Since the characteristic polynomial

$$P_A(\lambda) = \det(A - \lambda \mathbf{I}) = (-\lambda)^{n-1} \left(\frac{n(n+1)}{2} - \lambda \right),$$

$$\det(A + \mathbf{I}) = P_A(-1) = \frac{n(n+1)}{2} + 1.$$

10. (a) The velocity vector $(\dot{x}, \dot{y})$ is proportional to the gradient $\nabla c(x,y)$. Therefore $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\partial_y c}{\partial_x c} = \frac{2y}{x}$.

   (b) Solving the differential equation from part (a) gives $dy/2y = dx/x$, so $y = Kx^2$ with $K =$ const. Using the initial condition $y(100) = 100$ gives $K = 0.01$, so the answer is $y = f(x) = 0.01\, x^2$ (with $x \ge 0$).