

# Qualifying Exam

September 18, 2017

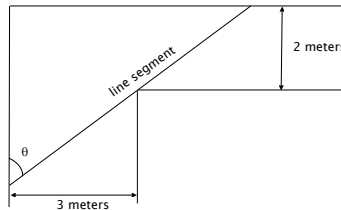
Number \_\_\_\_\_

EXPLAIN ALL YOUR ANSWERS AND SHOW ALL YOUR WORK.

NO CALCULATORS.

ALL ELECTRONIC DEVICES MUST BE TURNED OFF.

1. A corridor 3 meters wide meets a corridor 2 meters wide at a right angle (see the figure). A straight line segment is drawn on the floor so that its ends are on the outside walls and it touches the inside corner. Calculate the shortest length for such a line segment. For full credit, your final answer should be a number which does not use trigonometric functions or their inverses.



2. Let

$$D = \{(r, \theta) : r\theta \leq 1, 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

be a region in  $\mathbb{R}^2$  given in polar coordinates.

- (a) Sketch  $D$ .  
(b) Set up, but do not evaluate, iterated integrals to find the area of  $D$   
(i) with integration with respect to  $r$  first, and  
(ii) with integration with respect to  $\theta$  first.  
(c) Find the area of  $D$ .

3. The Fibonacci sequence  $(x_n)$  is defined by  $x_0 = x_1 = 1$  and

$$x_n = x_{n-1} + x_{n-2}, \quad n = 2, 3, \dots$$

- (a) Find a  $2 \times 2$  matrix  $A$  such that

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix}, \quad n = 2, 3, \dots$$

- (b) Find the eigenvalues and eigenvectors of  $A$ .  
(c) Find a closed-form expression for  $x_n$ ,  $n = 1, 2, \dots$ , i.e., an explicit formula for  $x_n$  as a function of  $n$ .

4. Let  $\mathbb{P}_3$  be the space of polynomials of degree 3 or less in the variable  $x$ . Recall that the standard basis for  $\mathbb{P}_3$  is  $\{1, x, x^2, x^3\}$ . Define  $T : \mathbb{P}_3 \rightarrow \mathbb{R}^2$  by

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix},$$

where  $p(a)$  means the polynomial  $p$  evaluated at  $x = a$ .

- (a) Prove that  $T$  is a linear transformation.  
(b) Find the standard matrix for  $T$  (i.e., relative to the standard bases).  
(c) Find a basis for the kernel (sometimes called the null space) of  $T$ .  
(d) Find the column space of the standard matrix for  $T$ .

- 5.(a) Describe in full detail how one can calculate the angle between the  $xy$ -plane and a tangent plane to a surface with equation  $z = h(x, y)$ .

- (b) Apply your method from part (a) to find all the points  $(x, y)$  such that the angle between the  $xy$ -plane and the tangent plane to the surface  $z = 10 - x^2 + 4x - y^2$  is  $\pi/4$ .

6. Let  $P$  be the region in  $\mathbb{R}^3$  given in Cartesian coordinates  $(x, y, z)$  by  $x \geq 0$ ,  $y \geq -1$ ,  $z \geq -2$ , and  $x + y + z \leq 3$ .

(a) Find the 4 vertices of  $P$ .

(b) Write down, but do not evaluate, one iterated integral whose value is the average value of  $z$  over the region  $P$ .

7. A manufacturer makes the same high quality bicycles in three facilities. The total cost, in dollars, in the first facility when  $q$  bicycles are made is  $c_1(q) = 2q^2 + 3q + 1000$ . The total cost, in dollars, in the second facility when  $q$  bicycles are made is  $c_2(q) = q^2 + 5q + 2000$ . The total cost, in dollars, in the third facility when  $q$  bicycles are made is  $c_3(q) = 3q^2 + q + 3000$ .

(a) Find the number of bicycles that should be made in each facility so that the total cost of manufacturing 2005 bicycles is minimized.

(b) Approximately how much does it cost to make an additional bicycle when 2005 bicycles are made?

8. For  $x > 0$ , let

$$g(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{x}{n}\right) \left(\frac{x n^2 + i x n}{n^2 + i^2 x^2}\right).$$

(a) Represent  $g(x)$  as an integral.

(b) Use your answer to part (a) to calculate  $\frac{dg}{dx}$ .

9. Evaluate exactly the following sums, justifying your work and giving your answers in the simplest form possible.

(a)  $\sum_{n=0}^{\infty} \left(\frac{1}{3^{n+1}} + \frac{2}{5^{n-1}}\right).$

(b)  $\sum_{n=3}^{\infty} \frac{1}{n^2 - n}.$

(c)  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{1}{2}\right)^n.$

10. A river flows into a reservoir at 20,000 cubic feet per second. The river water is 1 percent silt by volume. The reservoir has a fixed volume of  $4 \times 10^9$  cubic feet and starts out with no silt in it (at time zero). The silt mixes with the water in the reservoir so that 0.1 percent of the silt remains suspended in the water and the rest settles in the reservoir. The mixed water and silt flow out of the reservoir at 20,000 cubic feet per second.

(a) Write a differential equation that describes the evolution of the amount of silt in the reservoir over time.

(b) Calculate the amount of silt in the reservoir as a function of time.

(c) Either the reservoir fills with silt or the amount of silt in the reservoir approaches some value less than the volume of the reservoir, as time goes on. Decide which occurs and explain your decision.

(d) If you decided that the reservoir fills with silt, calculate the time at which this occurs. If you decided that the amount of silt approaches a value less than the volume of the reservoir, calculate the time at which the amount reaches 90 percent of the amount approached.