FALL 2017 GRADUATE QUALIFYING EXAM: ANSWERS AND HINTS

NOTE: For full credit, students are expected to provide COMPLETE solutions with all details.

1. The minimum length is $(2^{2/3}+3^{2/3})^{3/2}$. (*Hint*: The angle θ is marked in the figure, so use it! Find the length of the segment as a function of θ.)

2. (a) The curve $r = 1/\theta$ approaches the x-axis as $\theta \to 0$; it meets the circle $r = 1$ at a positive angle less than $\pi/2$ and the circle $r = 2$ at a smaller positive angle. (b) For (i), it is necessary to express the area as a sum of two iterated integrals, while for (ii) only one iterated integral is needed. (c) The area is 1, best computed using part (ii) of (b).

3. (a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. (b) Eigenvalues are $(1 \pm \sqrt{5})/2$ and eigenvectors $\begin{bmatrix} (1 \pm \sqrt{5})/2 \\ 1 \end{bmatrix}$. (c) An answer is

$$x_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

(*Hint*: Let $y_n = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$, so $y_n = Ay_{n-1}$, $n = 2, 3, \ldots$. Find c_1 and c_2 so that $y_1 = c_1 v_1 + c_2 v_2$, where v_1 and v_2 are the eigenvectors of A, and note that $y_n = c_1 \lambda_1^{n-1} v_1 + c_2 \lambda_2^{n-1} v_2$, where λ_1 and λ_2 are the corresponding eigenvalues. Alternatively, one can diagonalize A and write $A = PDP^{-1}$, so that $A^n = PD^nP^{-1}$, and find an equivalent answer that way.)

4. (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. (c) A basis is $\{-x + x^2, -x + x^3\}$. (Note that the basis vectors must be polynomials, not elements of $\mathbb{R}^4$.) (d) $\mathbb{R}^2$.

5. (a) The angle is

$$\arccos\left(1\Big/\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}\right) = \arctan\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}.$$

(*Hint*: The angle between the two planes equals the angle between vectors normal to the planes.) (b) Points on the circle $(x-2)^2 + y^2 = 1/4$ with center $(2, 0)$ and radius $1/2$.

6. (a) The vertices are $(0, 5, -2)$, $(0, -1, -2)$, $(6, -1, -2)$, and $(0, -1, 4)$. (b) The required integral is

$$\frac{1}{36}\int_0^6 \int_{-1}^{5-x} \int_{-2}^{3-x-y} z \, dz \, dy \, dx.$$

(*Hint*: P is a tetrahedron, so its volume is 1/3 times its base area times its height.)

7. (a) Let q_i be the number of bicycles made in the facility with cost c_i, $i = 1, 2, 3$. Then $q_1 = 547$, $q_2 = 1093$, and $q_3 = 365$. (*Hint*: Find the total cost as a function of the variables q_1, q_2, and q_3, and use Lagrange multipliers.) (b) Approximately \$2191, the corresponding value of the Lagrange multiplier λ.

8. (a) $g(x) = \int_0^x \frac{x+u}{1+u^2}\,du$. (*Hint*: Express the sum in $g(x)$ as a Riemann sum of a function $f(u)$ over the interval $0 \le u \le x$ with points $u_i = ix/n$, $i = 1, \ldots, n$, in its subdivision.) (b) $\frac{dg}{dx} = \frac{2x}{1+x^2} + \arctan x$.

9. (a) 13. (*Hint*: Split up the sum into two geometric series, explaining why this is valid.) (b) 1/2. (*Hint*: Telescoping series.) (c) ln(3/2). (*Hint*: Integrate the geometric series

$$1 + x + x^2 + \cdots = \frac{1}{1-x}, \qquad |x| < 1.)$$

10. (a) Let t denote time and let s denote the amount of silt in the reservoir. Then $\frac{ds}{dt} = 200 - \frac{s}{2 \cdot 10^8}$. (b) $s = 4 \cdot 10^{10}\left(1 - e^{-t/(2\cdot 10^8)}\right)$. (c) $s \to 4 \cdot 10^{10} > 4 \cdot 10^9$, so the reservoir fills up with silt. (d) $t = 2 \cdot 10^8 \ln(10/9)$.