# Qualifying Exam

Spring 2019
April 1, 2019

Number ____________________

You may use calculators for this exam. — Justify all your answers. Answer specific questions by giving the exact values, not approximations.

**Problem 1.** Consider the unit sphere. Determine the radius of a circular cylinder whose axis is along the sphere's diameter and which contains one-half of the sphere's volume. See Figure 1.

**Problem 2.** Find all the solutions of the equation

$$y'(x) = |y(x)|, \quad x \in \mathbb{R}$$

which are defined for all $x \in \mathbb{R}$.

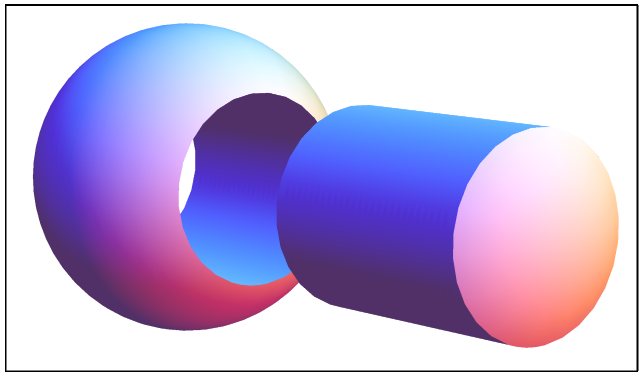

**Fig. 1:** The unit sphere cut by a cylinder

**Problem 3.** Figure 2 shows the parabola $y = x^2$ and a unit circle with its center on the $y$-axes. This unit circle intersects the parabola at the right angle. That means that at each point of intersection the tangent lines to the parabola and the unit circle are perpendicular. Find the center of this unit circle.

**Problem 4.** Let $P$ be a fixed point on the unit circle $\mathbb{T}$. Denote by $a_e$ the average of the Euclidian distances of all the points on $\mathbb{T}$ to the point $P$. (Two Euclidean distances are indicated as blue line segments in Figure 3.) Denote by $a_l$ the average of all shortest arc lengths from all the points on $\mathbb{T}$ to $P$. (Two shortest arc lengths are shown as teal circular arcs in Figure 3.)

(a) Use common sense about the averages to order the numbers 0, 1, 2, $a_e$, $a_l$ and $\pi$ in the increasing order. Explain your reasoning in as simple terms as possible without doing any calculus.

(b) Calculate $a_l$ exactly using integrals.

(c) Calculate $a_e$ exactly using integrals.

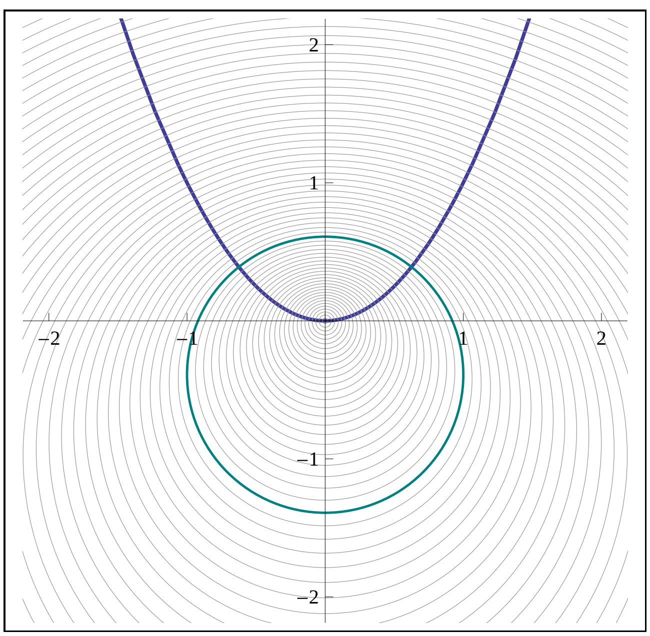


**Fig. 2:** $y = x^2$ and a unit circle

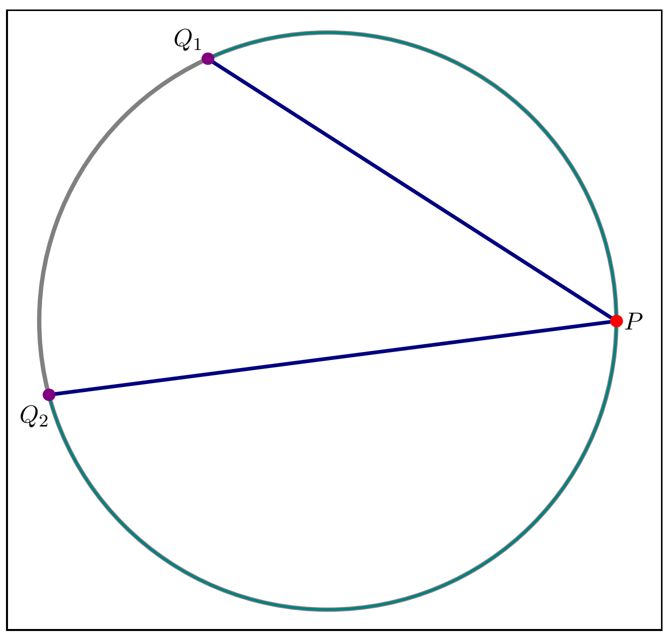


**Fig. 3:** The unit circle and averages

**Problem 5.** (a) Find the Maclaurin series for $(\sin x)^2$ using the Maclaurin series for $\cos(2x)$.

(b) Using the answer for part (a), find $\lim\limits_{x\to 0} \dfrac{(\sin x)^2 - x^2}{x^4}$.

**Problem 6.** Let $t \in \mathbb{R}$ and consider the functions $f_t : \mathbb{R} \to \mathbb{R}$ given by

$$f_t(x) = x + te^{-x^2} \quad \text{for all} \quad x \in \mathbb{R}.$$

(a) Prove that for each $t \in \mathbb{R}$ the function $f_t$ is a surjection.

(b) Prove that the function $f_t$ is a bijection if and only if $|t| \le \sqrt{e/2}$.

**Problem 7.** Let $A$ be a real $3\times 2$ matrix and let $B$ be a real $2\times 3$ matrix. Prove the following implication:

$$AB = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -3 & 2 \\ 2 & -4 & 3 \end{bmatrix} \qquad \Rightarrow \qquad BA = I_2$$

Hint: $AB$ is singular, $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ is an eigenvector of $AB$ and $AB$ has an eigenvalue of multiplicity 2.

**Problem 8.** Given

$$A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 2 & 2 & -3 \\ 2 & 3 & 3 & -5 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix},$$

find a vector $\mathbf{v} \in \operatorname{Nul} A$ and a vector $\mathbf{w} \in \operatorname{Row} A$ such that $\mathbf{y} = \mathbf{v} + \mathbf{w}$.
Hint that you need not use: Such vectors $\mathbf{v}$ and $\mathbf{w}$ are uniquely determined.

**Problem 9.** A classical calculus problem is as follows: Consider a piece of wire of length 4 and cut it in two pieces. Make one piece into a square and the other piece into a circle. Find the minimum and the maximum of the total area that is enclosed by such formed square and circle.
In this problem we ask you to solve the analogous problem for a cube and a sphere: Assume that the total surface area of a cube and a sphere is 4. Calculate the minimum and the maximum of the total volume enclosed by such a cube and a sphere.

**Problem 10.** Let $f : \mathbb{N} \to \mathbb{N}$ be a function where $\mathbb{N}$ denotes the set of positive integers.

(a) Prove that $\displaystyle\sum_{n=1}^{+\infty} \frac{f(n)}{n^2}$ diverges whenever $f$ is strictly increasing.

(b) Prove that there exists nondecreasing unbounded $f$ such that $\displaystyle\sum_{n=1}^{+\infty} \frac{f(n)}{n^2}$ converges.

OVER

*Solution of Problem* 1. Let $s \in (0,1)$. The volume of the part of the unit sphere which is inside the right circular cylinder of radius $s$ is

$$4\pi \int_0^s r\sqrt{1-r^2}dr = \frac{4}{3}\pi\left(1-\left(1-s^2\right)^{3/2}\right).$$

Since the volume of the unit sphere is $4\pi/3$, one-half of the unit sphere will be inside the cyinder with the radius $s = \sqrt{1-2^{-2/3}}$. □

*Solution of Problem* 2. We first look for the nonnegative solutions $y(x) \geq 0$. With this condition the given equation simplifies to $y'(x) = y(x)$. The general solution of this equation is $y(x) = C\exp(x)$ with $C \in \mathbb{R}$. Since we imposed the condition $y(x) \geq 0$, the solutions of the given equation are $y(x) = C\exp(x)$ with $C \geq 0$. Notice that all these solutions are defined on $\mathbb{R}$. Next, we look for the negative solutions of the given equation $y(x) < 0$. For such solutions the given equation simplifies to $y'(x) = -y(x)$. The general solution of the last equation is $y(x) = C\exp(-x)$ with $C \in \mathbb{R}$. However, since we imposed the condition $y(x) < 0$, the solutions of the given equation are $y(x) = C\exp(-x)$ with $C < 0$. Notice that all these solutions are defined on $\mathbb{R}$. In conclusion, the general solution of the given equation is the following family of functions

$$y(x) = Ce^x, \quad y(x) = 0, \quad y(x) = -Ce^{-x},$$

where $x \in \mathbb{R}$ and $C > 0$. □

*Solution of Problem* 3. Let the coordinates of the intersection point of the parabola and the unit circle be $(a, a^2)$. At any point of a circle, its radius and the tangent at that point are perpendicular. Therefore the tangent line to the parabola at the point $(a, a^2)$ goes through the center of the circle. Since the tangent line is

$$y = 2a(x-a) + a^2 = 2ax - a^2$$

and the center of the circle is on the $y$-axis, the circle center is $(0, -a^2)$. Since the radius of the circle is 1 we have that the distance of the points $(0, -a^2)$ and $(a, a^2)$ is 1. Solving $a^2 + 4a^4 = 1$ for $a > 0$ yields $a = \frac{1}{2}\sqrt{\frac{1}{2}\left(\sqrt{17}-1\right)}$. Consequently, the center of the pictured unit circle is $\left(0, (1-\sqrt{17})/8\right)$. □

*Solution of Problem* 4. (a) For a fixed point on the unit circle its Euclidean distance to $P$ is shorter than the length of the shorter arc from that point to $P$. Therefore $a_e < a_l$. By placing the two points which are at the Euclidean distance 1 from $P$ on the unit circle we see that the arc of points on $\mathbb{T}$ which are further than 1 away from $P$ is longer than the arc of points on $\mathbb{T}$ which are less than 1 from from $P$. Therefore, $1 < a_e$. Similar reasoning leads to $a_l < 2$. Hence

$$0 < 1 < a_e < a_l < 2 < \pi.$$

(b) This is the same as calculating the average distance of a point in the closed interval $(-\pi, \pi]$ to the origin, that is the piont 0. Thus

$$a_l = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|dx = \frac{\pi}{2}.$$

(c) Let $P = (1, 0)$. Let $Q = (\cos(x), \sin(x))$ with $x \in (-\pi, \pi]$. Then the Euclidean distance of $Q$ to $P$ is

$$\sqrt{(\sin x)^2 + (1-\cos x)^2} = 2\big|\sin(x/2)\big|.$$

Thus

$$a_e = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\big|\sin(x/2)\big|dx = \frac{2}{\pi}\int_0^{\pi}\sin(x/2)dx = \frac{4}{\pi}.$$ □

OVER

*Solution of Problem* 5. (a) We have that $(\sin x)^2 = \big(1-\cos(2x)\big)/2$ and the Maclaurin series for $\cos(2x)$ is

$$\cos(2x) = \sum_{k=0}^{+\infty} \frac{(-4)^k}{(2k)!}x^{2k} = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots .$$

Hence

$$(\sin x)^2 = \sum_{k=1}^{+\infty} 2\frac{(-4)^{k-1}}{(2k)!}x^{2k} = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \frac{1}{315}x^8 + \cdots .$$

(b) It follows from (a) that

$$(\sin x)^2 - x^2 = \sum_{k=2}^{+\infty} 2\frac{(-4)^{k-1}}{(2k)!}x^{2k} = x^4 \sum_{k=2}^{+\infty} 2\frac{(-4)^{k-1}}{(2k)!}x^{2(k-2)} = x^4\left(-\frac{1}{3} + \frac{2}{45}x^2 - \frac{1}{315}x^4 + \cdots\right).$$

This implies that $(\sin x)^2 - x^2 = x^4 g(x)$, where $g(x)$ is a continuous function defined on $\mathbb{R}$ with $g(0) = -1/3$. Consequently, the limit in (b) equals $-1/3$. □

*Solution of Problem* 6. (a) The claim is clear if $t = 0$. In the rest of this proof we assume that $t \neq 0$. Clearly,

$$-|t| \leq te^{-x^2} \leq |t| \quad \text{for all} \quad (t,x) \in \mathbb{R}\times\mathbb{R}.$$

Therefore

$$x - |t| \leq f_t(x) \leq x + |t| \quad \text{for all} \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{1}$$

Let $t \in \mathbb{R}\backslash\{0\}$ and $y \in \mathbb{R}$ be arbitrary. Set $x_1 = y - |t|$ and $x_2 = y + |t|$. Then, $x_1 < y < x_2$. By (1) we have

$$f_t(x_1) \leq x_1 + |t| = y = x_2 - |t| \leq f_t(x_2). \tag{2}$$

Since the function $f_t$ is clearly continuous on $\mathbb{R}$ and thus on $[x_1, x_2]$, by the Intermediate Value Theorem there exists $x \in [x_1, x_2]$ such that $f_t(x) = y$.
(b) We have

$$f_t'(x) = \frac{d}{dx}f_t(x) = 1 - 2txe^{-x^2} \quad \text{for all} \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{3}$$

The minimum of the derivative $f_t'(x)$ occurs at $x = \operatorname{sgn}(t)\sqrt{2}/2$ and that minimum is $1 - |t|\sqrt{2/e}$. Hence, if $|t| \leq \sqrt{e/2}$, then the function $f_t$ is strictly increasing; thus injective. For the converse, we prove its contrapositive. Assume that $|t| > \sqrt{e/2}$. Then the value of $f_t'(x)$ at $x = \operatorname{sgn}(t)\sqrt{2}/2$ is negative. Since the limit of $f_t'(x)$ as $x \to \infty$ is 1 and since $f_t'$ is a continuous function we conclude that there exists $x_0 \in \mathbb{R}$ such that $f_t'(x_0) = 0$ and $f_t'(x)$ changes sign at $x_0$. This implies that $f_t(x)$ has a local extremum at $x_0$. Since $f_t$ is continuous on $\mathbb{R}$ it cannot be an injection. □

*Solution of Problem* 7. We are given that the matrix $AB$ is singular. Hence 0 is an eigenvalue. The RREF of $AB$ is

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consequently the multiplicity of 0 as an eigenvalue of $AB$ is 1. Since $AB[1\ 1\ 1]^\top = [1\ 1\ 1]^\top$, 1 is the second eigenvalue of $AB$. The RREF of $AB - I_3$ is

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the multiplicity of the eigenvalue 1 is 2 and

$$AB\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix},\quad AB\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

The last two vector equalities imply that both vetors $[1\ 1\ 1]^\top$ and $[1\ 0\ -1]^\top$ belong to $\operatorname{Col} A$. Since these two vectors are linearly independent and $\dim \operatorname{Col} A \leq 2$ we conclude that $\operatorname{Col} A$ is spanned by $[1\ 1\ 1]^\top$ and $[1\ 0\ -1]^\top$. Since $AB$ acts as an identity on $[1\ 1\ 1]^\top$ and $[1\ 0\ -1]^\top$, it acts as an identity on $\operatorname{Col} A$. Consequently, $ABA = A$. Since the columns of $A$ are linearly independent the last equality yields $BA = I_2$. □

*Solution of Problem* 8. Since this problem asks about $\operatorname{Row} A$ and $\operatorname{Nul} A$, it is useful to find the basis of both of these spaces. For this aim RREF of $A$ is useful:

$$\begin{bmatrix}1&0&0&-1\\0&1&1&-1\\0&0&0&0\end{bmatrix}.$$

Thus the vectors $[1\ 0\ 0\ -1]^\top$ and $[0\ 1\ 1\ -1]^\top$ form a basis for $\operatorname{Row} A$, while the vectors $[0\ 1\ -1\ 0]^\top$ and $[1\ 1\ 0\ 1]^\top$ form a basis for $\operatorname{Nul} A$. Solving the vector equation

$$x_1\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}+x_2\begin{bmatrix}0\\1\\1\\-1\end{bmatrix}+x_3\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}+x_4\begin{bmatrix}1\\1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

for $x_1$, $x_2$, $x_3$, $x_4$ will lead to $\mathbf{v}$ and $\mathbf{w}$. So, row reduce

$$\begin{bmatrix}1&0&0&1&1\\0&1&1&1&2\\0&1&-1&0&3\\-1&-1&0&1&4\end{bmatrix}\quad\text{to}\quad\begin{bmatrix}1&0&0&0&-2\\0&1&0&0&1\\0&0&1&0&-2\\0&0&0&1&3\end{bmatrix}$$

to get

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=(-2)\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}+\begin{bmatrix}0\\1\\1\\-1\end{bmatrix}+(-2)\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}+3\begin{bmatrix}1\\1\\0\\1\end{bmatrix}=\begin{bmatrix}-2\\1\\1\\1\end{bmatrix}+\begin{bmatrix}3\\1\\2\\3\end{bmatrix}=\mathbf{w}+\mathbf{v}.$$

□

*Solution of Problem* 9. Denote by $x$ the side of a cube and by $y$ the radius of a sphere. Then the constraint in this problem is

$$6x^2+4\pi y^2=4,\quad x\geq 0,\quad y\geq 0.$$

Under this constraint we need to find the minimum and the maximum of the function

$$f(x,y)=x^3+\frac{4}{3}\pi y^3.$$

We use the method of Lagrange multipliers and, for positive $x$ and $y$, we set the system of equations:

$$\begin{aligned}3x^2&=\lambda(12x)\\4\pi y^2&=\lambda(8\pi y)\\6x^2+4\pi y^2&=4\end{aligned}$$

This system symplifies to

$$\begin{aligned} x &= 4\lambda \\ y &= 2\lambda \\ 3x^2 + 2\pi y^2 &= 2 \end{aligned}$$

Substituting the first two equations into the third one leads to the solution

$$\lambda = \frac{1}{2\sqrt{6+\pi}}, \quad x = \frac{2}{\sqrt{6+\pi}}, \quad y = \frac{1}{\sqrt{6+\pi}}$$

We also need to consider the points $(x, y)$ at the boundary of the constraint. That is

$$x = 0, \quad y = \frac{1}{\sqrt{\pi}} \quad \text{and} \quad x = \sqrt{\frac{2}{3}}, \quad y = 0.$$

The corresponding values of the function $f$, respectively, are

$$\frac{4}{3\sqrt{6+\pi}} \approx 0.440989, \quad \frac{4}{3\sqrt{\pi}} \approx 0.752253, \quad \left(\frac{2}{3}\right)^{\frac{3}{2}} \approx 0.544331.$$

Hence, the maximum volume occurs when there is no cube. The minimum volume occurs when the side of the cube is equal to the diameter of the sphere, both being $2/\sqrt{6+\pi}$ $\square$

*Solution of Problem* 10. (a) Assume that $f : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Since $f(1) \in \mathbb{N}$ we have $f(1) \geq 1$. Since $f$ is strictly increasing we have $f(2) > f(1)$. By the transitivity property of order $f(2) > f(1)$ and $f(1) \geq 1$ imply $f(2) > 1$. Since $f(2) \in \mathbb{N}$ we have $f(2) \geq 2$. Since $f$ is strictly increasing $f(3) > f(2)$. This strict inequality and $f(2) \geq 2$ yield $f(3) > 2$ and consequently $f(3) \geq 3$ as $f(3) \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $n > 3$. Repeating the preceding reasoning $n - 3$ more times, we conclude that $f(n) \geq n$. Thus for all $n \in \mathbb{N}$ we have $f(n) \geq n$. Therefore

$$\frac{f(n)}{n^2} \geq \frac{1}{n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Since the harmonic series diverges, by the comparison test the series in (a) also diverges.
(b) For arbitrary $n \in \mathbb{N}$ set

$$f(n) = k \quad \text{whenever} \quad 2^{k-1} \leq n < 2^k \quad \text{with} \quad k \in \mathbb{N}.$$

Or, equivalently,

$$f(n) = \lfloor \log_2(n) \rfloor + 1 \quad \text{for all} \quad n \in \mathbb{N}. \tag{4}$$

Then, for an arbitrary $k \in \mathbb{N}$ we have

$$\sum_{n=2^{k-1}}^{2^k-1} \frac{f(n)}{n^2} = k \sum_{n=2^{k-1}}^{2^k-1} \frac{1}{n^2} \leq k 2^{k-1} \frac{1}{2^{2(k-1)}} = \frac{k}{2^{k-1}}.$$

By the Ratio test the series $\sum_{k=1}^{+\infty} \frac{k}{2^{k-1}}$ converges. In fact $\sum_{k=1}^{+\infty} \frac{k}{2^{k-1}} = 4$. Therefore, for an arbitrary $p \in \mathbb{N}$ we have

$$\sum_{n=1}^{p} \frac{f(n)}{n^2} \leq \sum_{n=1}^{2^{\lfloor \log_2 p \rfloor}-1} \frac{f(n)}{n^2} \leq \sum_{k=1}^{\lfloor \log_2 p \rfloor} \frac{k}{2^{k-1}} \leq 4.$$

The last inequality proves that the sequence of the partial sums of the series $\sum_{n=1}^{+\infty} f(n)/n^2$ with $f$ defined in (4) is bounded above. Since this series has positive terms it follows that this series converges. As $f$ defined in (4) is nondecreasing and unbounded this proves (b). $\square$