

YOU MAY USE CALCULATORS FOR THIS EXAM. BE ADVISED, HOWEVER, THAT EVERY QUESTION CAN BE ANSWERED WITHOUT THE USE OF A CALCULATOR AND MORE THAN LIKELY CAN BE ANSWERED MORE EFFICIENTLY WITHOUT THE USE OF A CALCULATOR. — JUSTIFY ALL YOUR ANSWERS. ANSWER SPECIFIC QUESTIONS BY GIVING THE EXACT VALUES, NOT APPROXIMATIONS.

Problem 1. Consider the function

$$f(x) = \ln\left(\frac{1}{\sqrt{2 - e^{-2x}}}\right).$$

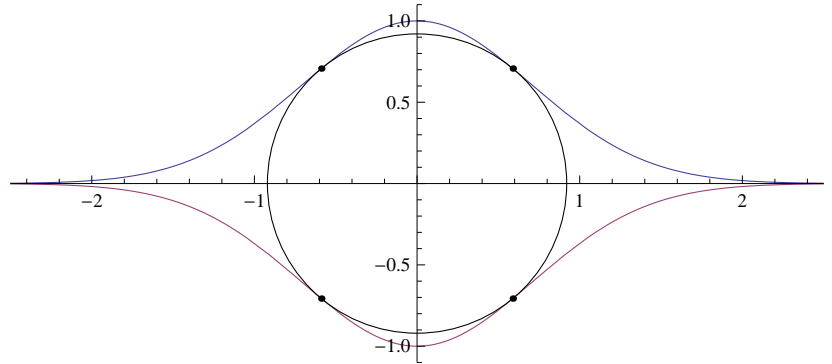
Determine the exact domain and range of this function. Prove that this function is an involution. Recall that a function $f : A \rightarrow A$ is an *involution* if f is a bijection and $f^{-1} = f$.

Problem 2. The figure on the right shows the functions

$$y = e^{-x^2} \quad \text{and} \quad y = -e^{-x^2}$$

and the circle centered at the origin that touches both graphs. Find the exact value of the radius of this circle.

The phrase “*circle touches a graph*” means that the circle and the graph have a common point at which they have a common tangent line.



Problem 3. Find all positive reals a for which the solution of the initial value problem

$$y' = \frac{1}{(1+t^2)y}, \quad y(0) = a$$

is defined for all $t \in \mathbb{R}$.

Problem 4. Consider the matrix $M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$

- Write a basis for the column space and a basis for the null-space of M .
- Calculate all eigenvalues of M .
- Is M diagonalizable?

Problem 5. Let $A = \begin{bmatrix} 5/2 & -1 \\ 3 & -1 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Calculate $\lim_{k \rightarrow \infty} A^k \mathbf{x}_0$.

Problem 6. Let $\{a_n\}$ be a sequence of positive real numbers.

- Prove or disprove: If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges.
- Prove or disprove: If $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- In addition assume that $\{a_n\}$ is nonincreasing. With this additional assumption, prove or disprove the above implications.

Hint: The inequality of arithmetic and geometric means can be very useful here. Write this inequality down before proceeding.

Problem 7. Find the volume of the solid given by

$$W = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leq 1, y^2 + z^2 \leq 1\}$$

by slicing the solid by planes parallel to the xy -plane. See Figure 1.

Problem 8. Figure 2 shows a right circular cone and a sphere completely contained in the cone. Find the radius of the base and the height of the right circular cone with the smallest surface area which completely contains the unit sphere. The mesh that you see in the picture is not part of the surface area of the cone; it is there just to indicate that the sphere is completely contained in the cone.

Problem 9. Recall that $[0, 1]^3$ denotes the unit cube in \mathbb{R}^3 , see Figure 3. That is the set

$$[0, 1]^3 = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \in [0, 1]\}.$$

Consider the function

$$f(x, y, z) = -(x \ln x + y \ln y + z \ln z), \quad (x, y, z) \in [0, 1]^3.$$

Notice that $\lim_{t \rightarrow 0^+} t \ln t = 0$. Therefore the function f is defined on all the sides of the unit cube.

- Determine the global maximum and the global minimum of the function f on the unit cube.
- Determine the global maximum and the global minimum of the function f on the intersection of the unit cube and the plane $x + y + z = 1$.

Problem 10. Let $r > 0$. Let W_r be the part of the unit sphere centered at the origin which is cut out by the cone $z = r\sqrt{x^2 + y^2}$. That is

$$W_r = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, r\sqrt{x^2 + y^2} \leq z\}.$$

Calculate r for which the volume of W_r equals one third of the volume of the unit sphere. See Figure 4 below.

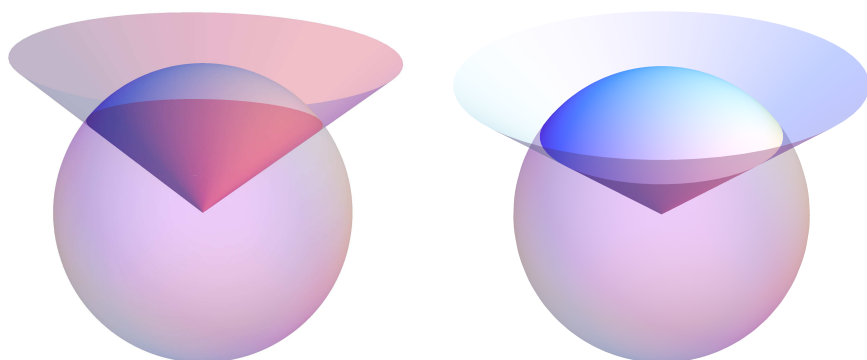


Figure 4: Two views of the region W_r

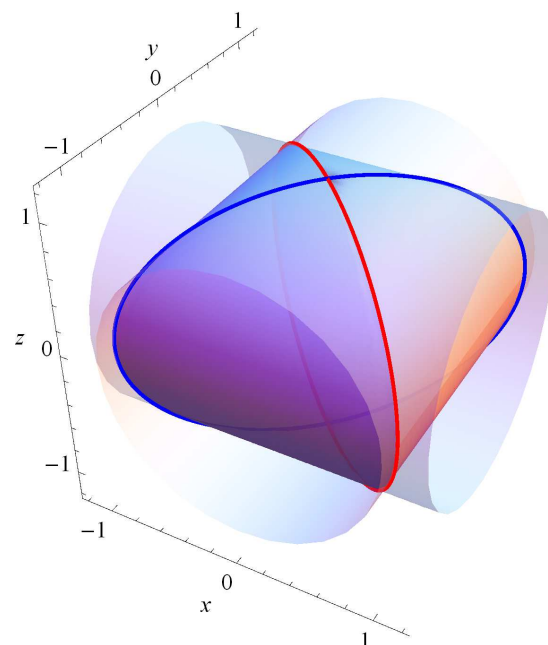


Figure 1: Two cylinders

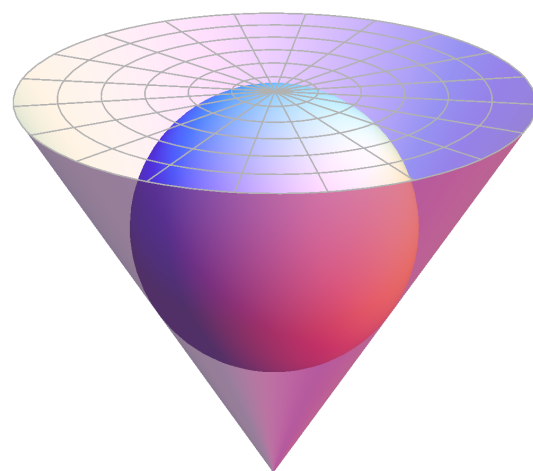


Figure 2: Sphere in a cone

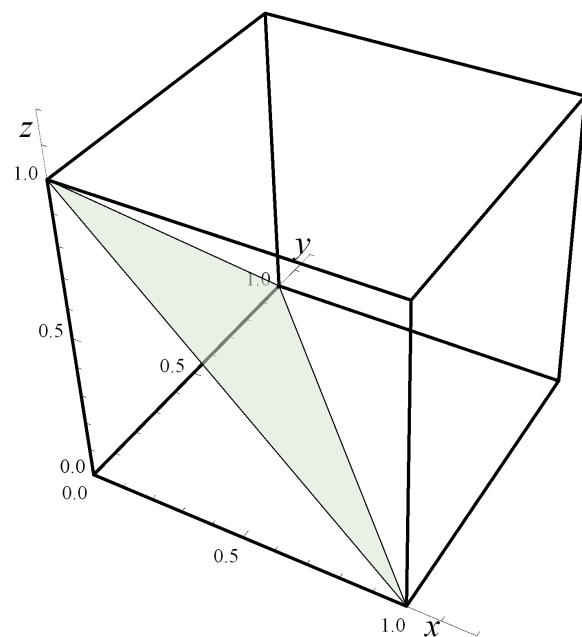


Figure 3: The unit cube